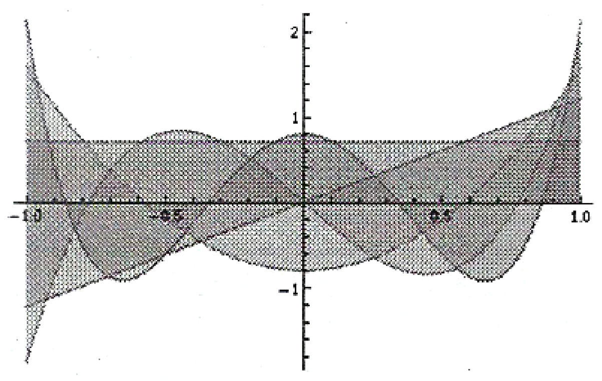
İZMİR UNIVERSITY OF ECONOMICS
Faculty of Engineering
EEE 281 Engineering Mathematics I
Fall 2023/2024



SOLUTIONS

FINAL EXAM
Jan 5, 2024
120 min

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Signature of Student:

Last Name :........................

Name :........................

Group :........................

Student No :........................

Question	Points	Page	Grade
1	15	2	
2	15	3	
3	20	4	
4	20	6	
5	15	7	
6	15	9	
Total	100		

Q1. (*15 pts*) Consider the following ordinary differential equation. Determine the solution corresponding to the given initial conditions.

$$y'' + 4\,y' + 20\,y = 0, \quad y(0) = 2, y'(0) = 8$$

Assume the solution as $y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t}, y'' = \lambda^2 e^{\lambda t}$.

Substiting into

$$(\lambda^2 + 4\lambda + 20)e^{\lambda t} = 0$$

Characteristic equation:

$$\lambda^2 + 4\lambda + 20 = 0$$

$$\Delta = 16 - 80 = -64$$

$$\lambda_{1,2} = \frac{-4 \mp 8i}{2} \begin{cases} -2+4i \\ -2-4i \end{cases}$$

Then by superposition

$$y = c_1' e^{\lambda_1 t} + c_2' e^{\lambda_2 t} = c_1' e^{(-2+4i)t} + c_2' e^{(-2t-4i)t}$$

$$= c_1' e^{-2t}(\cos 4t + i \sin 4t) + c_2' e^{-2t}(\cos 4t - i \sin 4t)$$

$$= e^{-2t}\left[\underbrace{(c_1' + c_2')}_{C_1}\cos 4t + \underbrace{(c_1' - c_2')i}_{C_2}\sin 4t\right]$$

$$y(t) = e^{-2t}(C_1 \cos 4t + C_2 \sin 4t) \Rightarrow y(0) = C_1 = 2 \Rightarrow C_1 = 2$$

$$y'(t) = -2e^{-2t}(C_1 \cos 4t + C_2 \sin 4t) + e^{-2t}(-4C_1 \sin 4t + 4C_2 \cos 4t)$$

$$y'(0) = (-2)(C_1) + (1)(4C_2) = 8 \Rightarrow -2C_1 + 4C_2 = 8$$

$$4C_2 = 8 + 4 \Rightarrow C_2 = 12/4$$

$$C_2 = 3$$

Therefore

$$y(t) = e^{-2t}(2\cos 4t + 3\sin 4t)$$

Q2. Consider the following differential equation where y is a function of x.

$$6xyy' = 3y^2 - 2x^2$$

(i) Put the above differential equation into a separable form through introducing a new variable u as

$$u = \frac{6y}{x}$$

(ii) Solve this separable differential equation for the given initial condition $y(2) = 0$.

(i) $6y = ux \Rightarrow 6y' = u'x + u$ (1)

$$6xyy' = 3y^2 - 2x^2 \rightarrow 6y' = 3\frac{y^2}{xy} - 2\frac{x^2}{xy} = 3\frac{y}{x} - 2\frac{x}{y}$$

$$6y' = \frac{1}{2}\left(\frac{6y}{x}\right) - 12\left(\frac{x}{6y}\right) = \frac{1}{2}u - \frac{12}{u}$$

$$6y' = \frac{u^2-24}{2u} \qquad (2)$$

From (1) & (2)

$$u'x + u = \frac{u^2-24}{2u} \Rightarrow u'x = \frac{u^2-24}{2u} - u = \frac{u^2-24-2u^2}{2u} = \frac{-u^2-24}{2u}$$

$$u'x = -\frac{u^2+24}{2u} \Rightarrow \frac{du}{dx}x = -\frac{u^2+24}{2u}$$

$$\frac{2u\,du}{u^2+24} = -\frac{dx}{x} \Rightarrow \int \frac{2u\,du}{u^2+24} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln(u^2+24) = -\ln x + c' \Rightarrow \ln(u^2+24) = \ln\frac{1}{x} + \ln c = \ln\left(\frac{c}{x}\right)$$

$$\Rightarrow u^2 + 24 = \frac{c}{x} \rightarrow \left(\frac{6y}{x}\right)^2 + 24 = \frac{c}{x}$$

$$\frac{36y^2}{x^2} + 24 = \frac{c}{x}$$

Multiply both sides by $x^2 \Rightarrow 36y^2 + 24x^2 = cx$

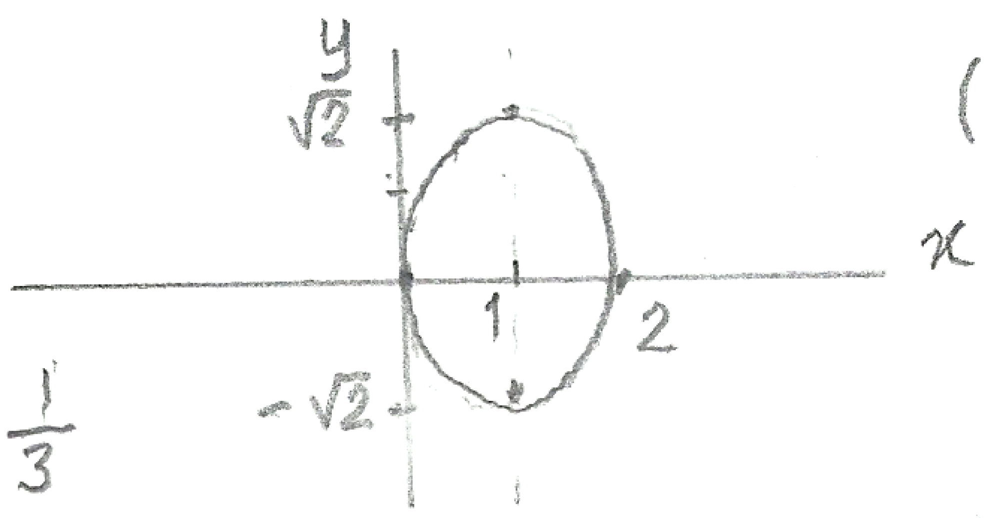
$x = 2 \Rightarrow y(2) = 0 \Rightarrow 0 + (24)(4) = c(2) \Rightarrow c = 48$

$$36y^2 + 24x^2 = 48x \Rightarrow 3y^2 + 2x^2 = 4x$$

$$3y^2 + 2(x^2 - 2x) = 0$$

$$3y^2 + 2(x^2 - 2x + 1) = 2$$

$$3y^2 + 2(x-1)^2 = 2 \Rightarrow \frac{y^2}{2} + \frac{(x-1)^2}{3} = \frac{1}{3}$$



$$\left(\frac{y}{\sqrt{2}}\right)^2 + \left(\frac{x-1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

Q3. Consider the following ordinary differential equation (y is a function of t)

$$y'' + 5y' + 4y = e^{-4t}$$

(i) Find the homogeneous solution

(ii) Determine the particular solution.

(iii) Determine the total solution corresponding to the initial conditions

$y(0) = 1, y'(0) = 4$

(i) Homogeneous solution

$$y'' + 5y' + 4y = 0$$

characteristic equation: $\lambda^2 + 5\lambda + 4 = 0$

$$\Delta = 25 - 16 = 9$$

$$\lambda_{1,2} = \frac{-5 \mp 3}{2} \begin{cases} -4 \\ -1 \end{cases}$$

Then the homogeneous solution:

$$y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-4t} + c_2 e^{-t}$$

(ii) Particular solution is required to be of the form e^{-4t}.
Since e^{-4t} exists in the homogeneous part, the assumption for the particular solution should be

$$y_p = Ate^{-4t}$$

$$y_p' = Ae^{-4t} - 4Ate^{-4t}$$

$$y_p'' = -4Ae^{-4t} - 4Ae^{-4t} + 16Ate^{-4t} = -8Ae^{-4t} + 16Ate^{-4t}$$

Substituting y_p'', y_p' and y_p in the given ODE:

$$-8Ae^{-4t} + 16Ate^{-4t} + 5(Ae^{-4t} - 4Ate^{-4t}) + 4Ate^{-4t} = e^{-4t}$$

$$\underbrace{(-8A + 5A)}_{=1} e^{-4t} + \underbrace{(16A - 20A + 4A)}_{=0} te^{-4t} = e^{-4t} \Rightarrow -3A = 1$$

$$A = -1/3$$

$$y_p = -\frac{1}{3} te^{-4t}$$

(iii) Total solution: $y_t = y_h + y_p = c_1 e^{-4t} + c_2 e^{-t} - \frac{1}{3} te^{-4t}$

$y_t(0) = 1 \Rightarrow c_1 + c_2 = 1$ (1)

$y_t'(0) = -4c_1 e^{-4t} - c_2 e^{-t} - \frac{1}{3} e^{-4t} + \frac{4}{3} te^{-4t} \Rightarrow y_t'(0) = -4c_1 - c_2 - \frac{1}{3} = 4$

$-4c_1 - c_2 = \frac{13}{3}$ (2)

From (1) & (2): $-3c_1 = \frac{16}{3} \Rightarrow c_1 = -\frac{16}{9} \Rightarrow c_2 = 1 - c_1 = \frac{25}{9}$

Then

$$y_t(t) = -\frac{16}{9} e^{-4t} + \frac{25}{9} e^{-t} - \frac{1}{3} te^{-4t}$$

Q4. Consider the following differential equation.

$$3y\,dx + 2x\,dy = 0$$

(i) Check whether the above differential equation is exact or not. Show the details of your work.

(ii) Determine the value of m in order the following function ~~is~~ an integrating factor to put the above differential equation into exact form

$$F(x,y) = y\,x^m$$

(iii) Find the solution for the initial value of $y(1) = 2$

(i) $\underbrace{3y}_{M}\,dx + \underbrace{2x}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 2$ $\Big\}$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so not exact !

(ii) Multiply each term by the integrating factor $F(x,y) = y\,x^m$.

$$\underbrace{3y^2 x^m}_{M}\,dx + \underbrace{2y\,x^{m+1}}_{N}\,dy = 0$$

$\frac{\partial M}{\partial y} = 6yx^m$, $\frac{\partial N}{\partial x} = 2(m+1)yx^m$ $\Big\}$ $6yx^m - 2(m+1)yx^m$

$6 = 2(m+1) \Rightarrow m = 2$

(iii) $M = 3y^2x^m = 3y^2x^2 \Rightarrow \frac{\partial u}{\partial x} = 3y^2x^2$

$u = \frac{3}{3}y^2x^3 + g(y) = y^2x^3 + g(y)$

$\frac{\partial u}{\partial y} = 2yx^3 + g'(y) = N = 2yx^3 \Rightarrow g'(y) = 0$

$g(y) = c'$

Then

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dx = 0 \Rightarrow d(u) = 0 \Rightarrow u = C$$

Hence

$$y^2x^3 + c' = C \rightarrow y^2x^3 = \underbrace{C - c'}_{C_{new}}$$

$$y^2x^3 = C_{new}$$

$x = 1 \Rightarrow y(1) = 2 \Rightarrow 4 \cdot 1 = C_{new} \Rightarrow C_{new} = 4$

The solution

$$y^2x^3 = 4 \Rightarrow y^2 = \frac{4}{x^3}$$

Q5. (*15 pts*) The paths C_1 and C_2 over which the following line integrals are to be calculated are as defined as:

- The path C_1 is the semicircle centered at (0,0) with radius r = 3. The path C_1 may be represented as C_1: $z = 3\,e^{i\theta}, -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$.
- The closed contour C_2 is the circle centered at (0,0) with radius r = 3. The closed contour C_2 may be represented as C_2: $z = 3\,e^{i\theta}, 0 \le \theta \le 2\pi$.

(i) Calculate the following integral over the path C_1

$$\int_{C_1} \left(\frac{1}{z}\right) dz$$

(ii) Calculate the following integral over the closed contour C_2

$$\oint_{C_2} \left(\frac{1}{z}\right) dz$$

(iii) Calculate the following integral over the path C_1

$$\int_{C_1} z\,dz$$

(iv) Calculate the following integral over the closed contour C_2

$$\oint_{C_2} z\,dz$$

(v) Comment on the results you obtained in part (ii) and (iv).

Over the circle with radius $r=3$,

$$z = 3e^{i\theta} \Rightarrow dz = 3i e^{i\theta} d\theta$$

(i) Over C_1 : $z = 3e^{i\theta}$ & $dz = 3ie^{i\theta} d\theta \quad -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$$\int_{C_1} \left(\frac{1}{z}\right) dz = \int_{-\pi/6}^{\pi/6} \left(\frac{1}{3e^{i\theta}}\right) 3ie^{i\theta} d\theta = \int_{-\pi/6}^{\pi/6} i\,d\theta = i\theta \Big|_{-\pi/6}^{\pi/6} = i\frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = i\frac{\pi}{3}$$

(ii) Over C_2 : $z = 3e^{i\theta}$, $dz = 3ie^{i\theta} d\theta \quad 0 \le \theta \le 2\pi$ (Counterclockwise)

$$\oint_{C_2} \left(\frac{1}{z}\right) dz = \int_0^{2\pi} \frac{1}{3e^{i\theta}} \cdot 3ie^{i\theta} d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$$

(iii)
$$\int_{C_1} z\,dz = \int_{-\pi/6}^{\pi/6} 3e^{i\theta} \cdot 3ie^{i\theta} = 9i \int_{-\pi/6}^{\pi/6} e^{i2\theta} d\theta = \frac{9i}{2} e^{i2\theta} \Big|_{-\pi/6}^{\pi/6}$$

$$= \frac{9i}{2}\left[e^{i\pi/3} - e^{-i\pi/3}\right] = -9 \frac{e^{i\pi/3} - e^{-i\pi/3}}{2i} = -9\sin\left(\frac{\pi}{3}\right) = -\frac{9\sqrt{3}}{2}$$

(iv)
$$\oint_{C_2} z\,dz = \int_0^{2\pi} 9i\,e^{i2\theta} d\theta = 9i\,e^{i2\theta} \Big|_0^{2\pi} = 9i(e^{i4\pi} - 1) = 9i(1-1) = 0$$

(v) By Cauchy's Integral Theorem

$\oint_{C_2} \left(\frac{1}{z}\right) dz = 2\pi i$ (singularity is at $z=0$); $\oint_{C_2} z\,dz = 0$ $f(z) = z$ is analytic.

Q6. (*15 pts*) Determine e^{At} if

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find eigenvalues and eigen vectors:

Eigenvalues:

$$\det |\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 1$$

Eigen vectors:

For $\lambda_1 = 2$

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0} \Rightarrow (\underline{A} - 2\underline{I})\underline{x} = 0$$

$$\begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \quad \begin{matrix} 3x_2 = 0 \\ x_1 \text{ any value} \end{matrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 1$

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0} \Rightarrow (\underline{A} - \underline{I})\underline{x} = 0$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{matrix} x_1 + 3x_2 = 0 \\ x_2 = -1, \ x_1 = 3 \end{matrix} \Rightarrow \underline{x}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$e^{\underline{A}t} = \underline{X} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \underline{X}^{-1}$$

where

$$\underline{X} = [\underline{x}_1 \ \underline{x}_2] = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$\underline{X}^{-1} = \frac{1}{\det(\underline{X})}\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix} = \frac{1}{(-1)}\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$e^{\underline{A}t} = \underline{X}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix}\underline{X}^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 3e^{2t} \\ 0 & -e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 3(e^{2t} - e^{t}) \\ 0 & e^{t} \end{bmatrix}$$