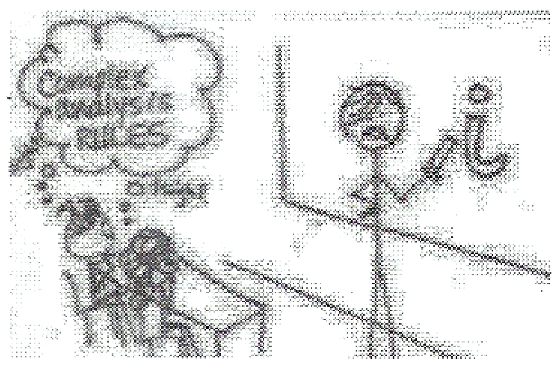
SOLUTIONS

İZMİR UNIVERSITY OF ECONOMICS
Faculty of Engineering
EEE 281 Engineering Mathematics I
Fall 2023/2024



MIDTERM EXAM 1
Nov 5, 2023
120 min

Information on exam rules

Electronic devices such as laptops, mobile phones, and smartwatches are generally prohibited in the examination room. However, exceptions can be made for individuals with special needs, provided they have valid medical documentation. Requests for exceptions must be submitted with prior written approval from the academic advisor, and they should include details on the necessary measures to maintain the integrity and security of the examination.

Please refrain from engaging in cheating or any other prohibited activities during the examination. Suspected cheating may result in a score of zero on your exam, and any students found cheating may face disciplinary actions in accordance with law #2547. This includes actions such as using unauthorized electronic devices, communicating with classmates, exchanging exam or formula sheets, or using unauthorized written materials during the exam, all of which qualify as attempted cheating.

Declaration

I affirm that the activities and assessments completed as part of this examination are entirely my own work and comply with all relevant rules regarding copyright, plagiarism, and cheating. I acknowledge that if there is any question regarding the authenticity of any portion of my assessment, I may be subject to oral examination. The signatory of evidence records may also be contacted, or a disciplinary process may be initiated as per law #2547.

Signature of Student:

Last Name :......................

Name :......................

Group :......................

Student No :......................

Question	Points	Grade
1	30	
2	25	
3	20	
4	25	
TOTAL	100	

Useful Trigonometric Values	
$cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$
$cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Q1. (30 pts)

(i) (7 pts) Determine the real and the imaginary parts of z where

$$z = \frac{-2-3i}{4+3i}$$

(ii) (8 pts) Determine the real and the imaginary parts of z where

$$z = \sqrt{1-i\sqrt{3}}$$

(iii) (7 pts) (a) Determine the roots of the following equation. (b) Verify your results.

$$-4z^2 + 4iz + 1 = 0$$

(iv) (8 pts) Determine the root(s) of the following equation.

$$\sqrt{z} + i\sqrt{z} = \sqrt{2}\, e^{i(3\pi/4)}$$

(i) $z = \dfrac{-2-3i}{4+3i} = \dfrac{-2-3i}{\underset{(4-3i)}{4+3i}} = \dfrac{(-2-3i)(4-3i)}{16+9} = \dfrac{-8-9+i(-12+6)}{25} = \dfrac{-17-6i}{25}$

$Re(z) = -\frac{17}{25}$; $Im(z) = -\frac{6}{25}$

(ii) $z = \sqrt{1-i\sqrt{3}}$

$1-i\sqrt{3} = re^{i\theta} \rightarrow r = \sqrt{1+3} = 2$, $\theta = \tan^{-1}(-\sqrt{3}) = -\pi/3$

$1-i\sqrt{3} = 2e^{-i\pi/}$

$z = \sqrt{1-i\sqrt{3}} = (1-i\sqrt{3})^{1/2} = (2e^{-i\pi/3})^{1/2} = (2)^{1/2} e^{-i\pi/6}$

$= \sqrt{2}\left[\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right] = \sqrt{2}\left[\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6})\right]$

$= \sqrt{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{\sqrt{3}}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \Rightarrow$

$Re(z) = \sqrt{3}/\sqrt{2}$, $Im(z) = -1/\sqrt{2}$

(iii) (a) $-4z^2 + 4iz + 1 = 0$

Quadratic equation with $a = -4$, $b = 4i$, $c = 1$

$z_{1,2} = \dfrac{-b \mp \sqrt{b^2-4ac}}{2a} = \dfrac{-4i \mp \sqrt{-16+16}}{-8} = \dfrac{-4i}{-8} = \frac{1}{2}i$

(b) Verification $z = i/2$, $z^2 = -1/4$

$-4z^2 + 4iz + 1 = (-4)(-\frac{1}{4}) + (4i)(-\frac{1}{2}i) + 1$

$= 1 - 2 + 1 = 0$

On the other hand,

$-4z^2 + 4iz + 1 = (2iz)^2 + 4iz + 1 = (2iz+1)^2 = 0 \Rightarrow 2iz = -1$

$z = -1/2i = i/2$

(iv) $\sqrt{z} + i\sqrt{z} = \sqrt{2}\, e^{i(3\pi/4)}$

$$(1+i)\sqrt{z} = \sqrt{2}\, e^{i(3\pi/4)} \Rightarrow \sqrt{z} = \frac{\sqrt{2}\, e^{i(3\pi/4)}}{1+i}$$

$$1+i = re^{i\theta} \Rightarrow r = \sqrt{1+1} = \sqrt{2}\,,\ \theta = \tan^{-1} 1 = \pi/4 \rightarrow 1+i = \sqrt{2}\, e^{i\pi/4}$$

Then $\sqrt{z} = \dfrac{\sqrt{2}\, e^{i(3\pi/4)}}{\sqrt{2}\, e^{i\pi/4}} = e^{i\pi/2}$.

$$z^{1/2} = e^{i(\frac{\pi}{2} + 2\pi n)} \Rightarrow z = e^{i2(\frac{\pi}{2} + 2\pi n)} = e^{i(\pi + 4\pi n)}$$

$$= \cos(\pi + 4\pi n) + i \underbrace{\sin(\pi + 4\pi n)}_{=0} = -1$$

verification

$$\sqrt{z} + i\sqrt{z} = \sqrt{2}\, e^{i(3\pi/4)}$$

$$\sqrt{-1} + i\sqrt{-1} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$i + i^2 = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$-1+i = -1+i \ \checkmark$$

Q2. (25 pts)
(i) (10 pts) Determine a so that the function given below is harmonic.

$$v(x,y) = a\,x^3 + x\,y$$

(ii) (15 pts) Find its harmonic conjugate using the value of a found in part (i).

(i) $v(x,y) = ax^3 + xy \Rightarrow v_x = 3ax^2 + y \rightarrow v_{xx} = 6ax$ Laplace equation
$v_y = x \rightarrow v_{yy} = 0$ $v_{xx} + v_{yy} = 0$
$6ax + 0 = 0 \rightarrow a = 0$

Then $v(x,y) = xy$

(ii) To determine harmonic conjugate, we must use Cauchy-Riemann equations.
$u_x = v_y$ & $u_y = -v_x$
Since $v(x,y) = xy$ is known, our job is to determine $u(x,y)$.
Then from the above part (i):

$u_x = v_y = x$ (1)

$u_y = -v_x = -y$ (2)

From (1):
$u_x = x \Rightarrow u = \int x\,\partial x = \frac{x^2}{2} + h(y)$

$u_y = h'(y)$ (3)

From (2) & (3)
$u_y = -y = h'(y) \Rightarrow h(y) = -\frac{y^2}{2} + c \Rightarrow u(x,y) = \frac{x^2}{2} - \frac{y^2}{2} + c$

Hence
$u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + c$ & $v(x,y) = xy$

and
$f(z) = u(x,y) + i\,v(x,y)$

Q3. (20 pts) Calculate the following integral

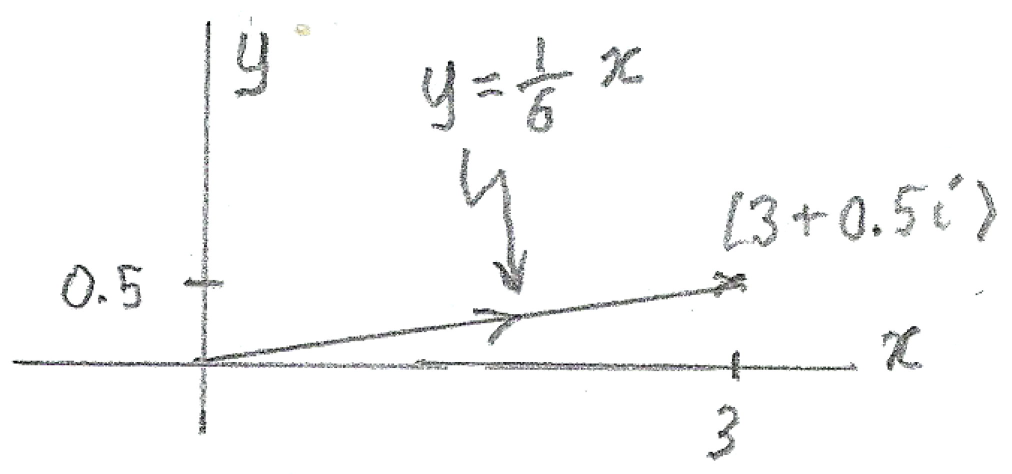
$$\int_C \left(\frac{18}{\pi}\right) dz$$

over the paths defined below. (Note that $z = x + iy$)

(i) (10 pts) C: The shortest path from $0 + 0i$ to $3 + 0.5i$.

(ii) (10 pts) C: The sine function $y = \sin\left(\frac{\pi}{18}x\right)$ from $0 + 0i$ to $3 + 0.5i$.

(i)



Over the path $y = mx$

$x = 3, y = 0.5 \Rightarrow 0.5 = m3 \Rightarrow m = \frac{0.5}{3} = \frac{1}{6}$

Then over the path

$y = \frac{1}{6}x$

$z = x + iy$

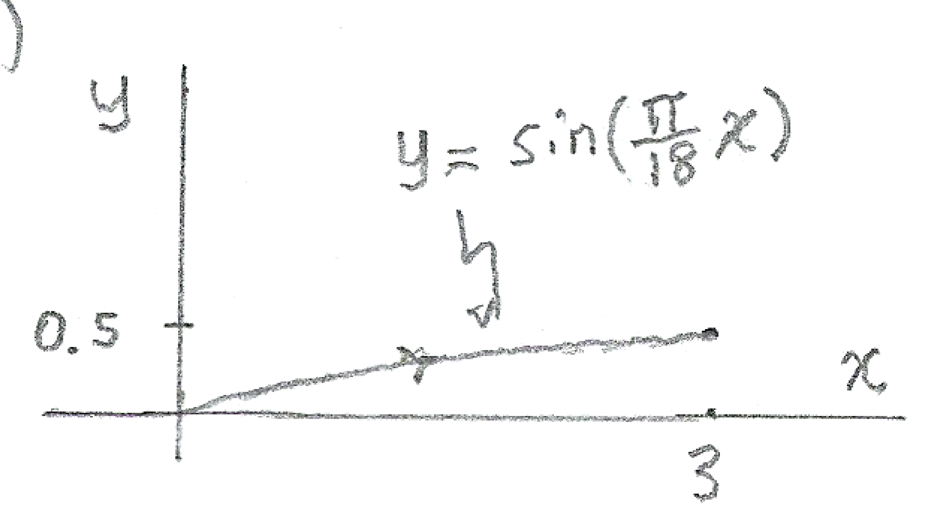
Let $x = t$ & $y = \frac{1}{6}x = \frac{1}{6}t \rightarrow z = t + i\frac{1}{6}t = (1 + \frac{1}{6}i)t \quad 0 \leq t \leq 3$

$dz = (1 + \frac{1}{6}i)dt$

$$\int_C \frac{18}{\pi} dz = \frac{18}{\pi}\int_C dz = \frac{18}{\pi}\int_0^3 (1 + \frac{1}{6}i)dt = \frac{18}{\pi}(1 + \frac{1}{6}i)\int_0^3 dt$$

$$= \frac{18}{\pi}(1 + \frac{i}{6})\, t\Big|_0^3 = \frac{18 \cdot 3}{\pi}(1 + \frac{i}{6}) = \frac{9}{\pi}(6 + i) = \frac{54}{\pi} + i\frac{9}{\pi}$$

(ii)



Over the path $y = \sin\frac{\pi}{18}x$

$z = x + iy$

Let $x = t, y = \sin\frac{\pi}{18}t \quad 0 \leq t \leq 3$

Over the path $z = t + i\sin(\frac{\pi}{18}t) \quad 0 \leq t \leq 3$

$dz = [1 + \frac{\pi}{18}i\cos(\frac{\pi}{18}t)]\, dt \quad 0 \leq t \leq 3$

$$\int_C \frac{18}{\pi} dz = \frac{18}{\pi}\int_C dz = \frac{18}{\pi}\int_0^3 [1 + \frac{\pi}{18}i\cos(\frac{\pi}{18}t)]dt = \frac{18}{\pi}[t + i\sin(\frac{\pi}{18}t)]_0^3$$

$$= \frac{18}{\pi}[3 + i\sin(\frac{\pi}{6})] = \frac{18}{\pi}[3 + i\frac{1}{2}] = \frac{54}{\pi} + i\frac{9}{\pi}$$

Q4. (25 pts) Assume $z_1 = 3$ and $z_2 = 3\,i$

(i) (5 pts) Evaluate the following integral for any closed contour enclosing only the point $z_1 = 3$.

$$\oint_C \frac{z}{(z-z_1)(z-z_2)}\,dz \qquad\qquad (z_1 = 3,\ z_2 = 3\,i)$$

(ii) (5 pts) Evaluate the above integral for any closed contour enclosing only the point $z_2 = 3\,i$.

(iii) (10 pts) Evaluate the above integral for any contour enclosing both the points z_1 and z_2.

(iv) (5 pts) Evaluate the above integral for any contour not containing the points z_1 and z_2.

(i) $\oint_C \frac{z}{(z-z_1)(z-z_2)}\,dz = \oint_C \frac{z/(z-z_2)}{z-z_1}\,dz = 2\pi i f(z_1)$ (for a contour containing z_1)

$= 2\pi i f(z_1)$ where $f(z) = \frac{z}{z-z_2}$ (Cauchy's integral)

$= 2\pi i \frac{z_1}{z_1-z_2} = 2\pi i \frac{3}{3-3i} = \frac{2\pi i}{1-i} = 2\pi i \frac{1+i}{1+1} = \pi i(1+i)$

$= -\pi + \pi i$

(ii) $\oint_C \frac{z}{(z-z_1)(z-z_2)}\,dz = \oint \frac{z/(z-z_1)}{z-z_2}\,dz = 2\pi i f(z_2)$ (for a contour containing z_2)

where $f(z) = \frac{z}{z-z_1} \Rightarrow f(z_2) = \frac{z_2}{z_2-z_1} = \frac{3i}{3i-3} = \frac{i}{-1+i} = \frac{i(-1-i)}{2}$

$\oint_C \frac{z}{(z-z_1)(z-z_2)}\,dz = 2\pi i f(z_2) = 2\pi i \frac{i(-1-i)}{2} = \pi(1+i)$

(iii) $\oint_C \frac{z}{(z-z_1)(z-z_2)}\,dz = ?$

Since the Cauchy's Integral formula applies over a single singularity, above integrand should be decomposed using partial fractions.

$$\frac{z}{(z-z_1)(z-z_2)} = \frac{A}{z-z_1} + \frac{B}{z-z_2} = \frac{A(z-z_2)+B(z-z_1)}{(z-z_1)(z-z_2)} = \frac{(A+B)z - Az_2 - Bz_1}{(z-z_1)(z-z_2)}$$

Then $A+B = 1$ (1)

$-Az_2 - Bz_1 = 0$ (2)

From (2): $-Az_2 = Bz_1 \Rightarrow B = -\frac{z_2}{z_1}A$ (3)

Substituting (3) into (1):

$A - \frac{z_2}{z_1}A = 1 \Rightarrow \frac{A(z_1-z_2)}{z_1} = 1 \Rightarrow A = \frac{z_1}{z_1-z_2}$

$B = -\frac{z_2}{z_1}A = -\frac{z_2}{z_1-z_2}$

Hence

$$\oint_C \frac{z}{(z-z_1)(z-z_2)}dz = \oint_C \frac{z_1/(z_1-z_2)}{z-z_1}dz + \oint_C \frac{-z_2/(z_1-z_2)}{z-z_2}dz$$

Then we can use Cauchy's Integral formula for each term.

$$\oint_C \frac{z}{(z-z_1)(z-z_2)}dz = 2\pi i f_1(z_1) + 2\pi i f_2(z_2)$$

where

$$f_1(z) = \frac{z_1}{z_1-z_2} \Rightarrow f_1(z_1) = \frac{z_1}{z_1-z_2}$$

and

$$f_2(z) = -\frac{z_2}{z_1-z_2} \Rightarrow f_2(z_2) = -\frac{z_2}{z_1-z_2}$$

So

$$\oint_C \frac{z}{(z-z_1)(z-z_2)}dz = 2\pi i \frac{z_1}{z_1-z_2} - 2\pi i \frac{z_2}{z_1-z_2} = 2\pi i \left(\frac{z_1-z_2}{z_1-z_2}\right)$$

$$= 2\pi i$$

(iv) Since z_1 and z_2 are not inside the contour, $f(z)$ is analytic over the domain enclosed by C; the integral of $f(z)$ over such a contour is 0. (Cauchy's Integral formula)

$$\oint_C \frac{z}{(z-z_1)(z-z_2)}dz = 0$$