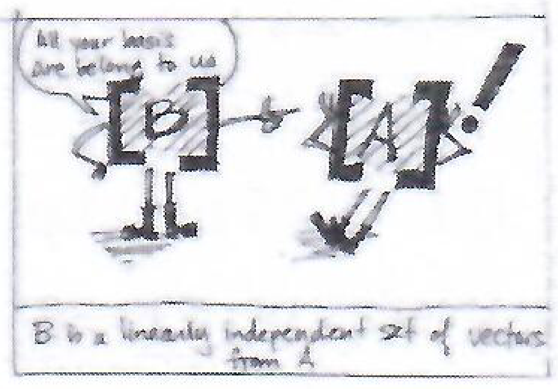
SOLUTIONS



İZMİR UNIVERSITY OF ECONOMICS
Faculty of Engineering
EEE 281 Engineering Mathematics I
Fall 2023/2024

MIDTERM EXAM 2
Dec 3, 2023
120 min

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Signature of Student:

	Question	Points	Grade
Last Name :........................	1	25	
Name :........................	2	25	
Group :........................	3	25	
Student No :........................	4	25	
	TOTAL	100	

Q1. (*25 pts*) Let

$$A=\begin{bmatrix}-1 & 2 & 1\\ 4 & 1 & 3\\ 2 & -1 & 2\end{bmatrix},\ B=\begin{bmatrix}1 & 3\\ 2 & -1\\ 2 & -1\end{bmatrix},\ C=\begin{bmatrix}-1 & 2\\ 1 & 4\end{bmatrix},\ D=\begin{bmatrix}2\\ 2\\ 2\end{bmatrix},\ E=\begin{bmatrix}3\\ 2\\ 2\end{bmatrix}$$

Find the following expressions or give reasons why they are not defined.

(i) $\boldsymbol{A\,B}$ (iv) $\boldsymbol{D\,E^T}$

(ii) $\boldsymbol{A\,C}$ (v) $\boldsymbol{C^{-1}}$

(iii) $\boldsymbol{B^T A}$

(i)

$$\underline{A}\,\underline{B}=\underbrace{\begin{bmatrix}-1 & 2 & 1\\ 4 & 1 & 3\\ 2 & -1 & 2\end{bmatrix}}_{3\times 3\ \checkmark}\underbrace{\begin{bmatrix}1 & 3\\ 2 & -1\\ 2 & -1\end{bmatrix}}_{3\times 2}=\underbrace{\begin{bmatrix}5 & -6\\ 12 & 8\\ 4 & 5\end{bmatrix}}_{3\times 2}$$

(ii)

$$\underline{A}\,\underline{C}=\underbrace{\begin{bmatrix}-1 & 2 & 1\\ 4 & 1 & 3\\ 2 & -1 & 2\end{bmatrix}}_{3\times 3}\underbrace{\begin{bmatrix}-1 & 2\\ 1 & 4\end{bmatrix}}_{2\times 2}=\text{not defined !}$$

$3\times3 \;\times\; 2\times2$

Not same

(iii)

$$\underline{B}^T\underline{A}=\underbrace{\begin{bmatrix}1 & 2 & 2\\ 3 & -1 & -1\end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix}-1 & 2 & 1\\ 4 & 1 & 3\\ 2 & -1 & 2\end{bmatrix}}_{3\times 3}=\underbrace{\begin{bmatrix}11 & 2 & 11\\ -9 & 6 & -2\end{bmatrix}}_{2\times 3}$$

$\checkmark$

(iv)

$$\underline{D}\,\underline{E}^T=\underbrace{\begin{bmatrix}2\\ 2\\ 2\end{bmatrix}}_{3\times 1}\underbrace{\begin{bmatrix}3 & 2 & 2\end{bmatrix}}_{1\times 3}=\underbrace{\begin{bmatrix}6 & 4 & 4\\ 6 & 4 & 4\\ 6 & 4 & 4\end{bmatrix}}_{3\times 3}$$

(iv)

$$\underline{C}=\begin{bmatrix}-1 & 2\\ 1 & 4\end{bmatrix}\qquad \det(C)=-4-2=-6$$

$$\underline{C}^{-1}=\frac{1}{(-6)}\begin{bmatrix}4 & -2\\ -1 & -1\end{bmatrix}=\begin{bmatrix}-2/3 & 1/3\\ 1/6 & 1/6\end{bmatrix}$$

Check:

$$\underline{C}\,\underline{C}^{-1}=\begin{bmatrix}-1 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}-2/3 & 1/3\\ 1/6 & 1/6\end{bmatrix}=\begin{bmatrix}\frac{2}{3}+\frac{2}{6} & -\frac{1}{3}+\frac{2}{6}\\ -\frac{2}{3}+\frac{4}{6} & \frac{1}{3}+\frac{4}{6}\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Q2. (*25 pts*) Consider the linear system of equations given below.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & -1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ 12 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

(i) For each part, obtain the corresponding augmented matrix and solve the unknowns by Gauss Elimination method.

(ii) Solve part (b) using Cramer's Rule and compare your results you obtained in part (i).

(i) $\tilde{\underline{A}} = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 11 \\ -1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 12 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 11 \\ 0 & 5 & 1 & 12 \\ 0 & -5 & 0 & -10 \end{array}\right]$

$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 11 \\ 0 & 5 & 1 & 12 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Then $x_3 = 2$

$5x_2 + x_3 = 12 \to 5x_2 = 12-2 = 10 \to x_2 = 2$

$x_1 + 3x_2 + 2x_3 = 11 \to x_1 = 11-6-4 \to x_1 = 1$

$\Bigg\} \quad \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\tilde{\underline{B}} = \left[\begin{array}{cc|c} 2 & 4 & -4 \\ 3 & 7 & -8 \end{array}\right] \xrightarrow{3R_1-2R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 4 & -4 \\ 0 & -2 & 4 \end{array}\right]$

Therefore

$-2x_2 = 4 \to x_2 = -2$

$2x_1 + 4x_2 = -4 \to 2x_1 = -4+8 = 4 \to x_1 = 2$

$\Big\} \quad \underline{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

(ii)

$\det(\underline{B}) = \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 14-12 = 2$

$x_1 = \dfrac{\begin{vmatrix} -4 & 4 \\ -8 & 7 \end{vmatrix}}{\det(\underline{B})} = \dfrac{-28-(-32)}{2} = \dfrac{4}{2} = 2$

$x_2 = \dfrac{\begin{vmatrix} 2 & -4 \\ 3 & -8 \end{vmatrix}}{\det(\underline{B})} = \dfrac{-16-(-12)}{2} = \dfrac{-4}{2} = -2$

$\Bigg\} \quad \underline{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ ✓

[redacted]

Q3. (*25 pts*) Consider the given matrix $\boldsymbol{A}$ and one of its eigenvectors $\boldsymbol{x}_1$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 2 & 0 \\ 0 & 4 & -5 \end{bmatrix}, \qquad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

(i) Show that $\boldsymbol{x}_1$ is an eigenvector and determine the corresponding eigenvalue.
(ii) Determine the other eigenvalues.

(i) If $\underline{x}_1$ is an eigenvector, then it should satisfy the following

$$\underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 2 & 0 \\ 0 & 4 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} = \underset{\lambda_1}{(3)}\underset{\underline{x}_1}{\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}} \Rightarrow$$ The corresponding eigenvalu then $\lambda_1 = 3$

(ii) The characteristic equation:

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 4 & 2-\lambda & 0 \\ 0 & 4 & -5-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 4 & -5-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & 1 \\ 4 & -5-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(-5-\lambda) - 4(-4) = 0$$

$$= (1-\lambda)(\lambda^2 + 3\lambda - 10) + 16 = 0$$

$$= -\lambda^3 + \lambda^2 - 3\lambda^2 + 3\lambda + 10\lambda - 10 + 16 = 0$$

$$= -\lambda^3 - 2\lambda^2 + 13\lambda + 6 = 0$$

To determine the roots, the characteristic equation should be factorized. Since $\lambda = 3$ is a root, $(\lambda - 3)$ is one the factor. To determine the other factor, one can use one of the following methods:

(i) Through long division

$$\begin{array}{l|l} -\lambda^3 - 2\lambda^2 + 13\lambda + 6 & \lambda - 3 \\ \hline \pm\lambda^3 \mp 3\lambda^2 & -\lambda^2 - 5\lambda - 2 \\ \hline \quad -5\lambda^2 + 13\lambda + 6 & \\ \quad \pm 5\lambda^2 \mp 15\lambda & \\ \hline \qquad -2\lambda + 6 & \\ \qquad -2\lambda + 6 & \\ \hline \qquad\quad 0 & \end{array}$$

Then $|\underline{A} - \lambda \underline{I}| = -\lambda^3 - 2\lambda^2 + 13\lambda + 6 = (\lambda - 3)(-\lambda^2 - 5\lambda - 2) = 0$

(ii) Predicting a form for the second factor as $A\lambda^2+B\lambda+C$. Then

$$-\lambda^3-2\lambda^2+13\lambda+6=(\lambda-3)(A\lambda^2+B\lambda+C)$$

It is easily obtained that $\boxed{A=-1}$, then (coeffient of λ^3 is -1)

$$-\lambda^2-2\lambda^2+13\lambda+6=(\lambda-3)(-\lambda^2+B\lambda+C)$$

$$\lambda=0 \Rightarrow \quad 6=(-3)(C) \Rightarrow \boxed{C=-2}$$

$$-\lambda^2-2\lambda^2+13\lambda+6=(\lambda-3)(-\lambda^2+B\lambda-2)$$

$$\lambda=1 \Rightarrow -1-2+13+6=(1-3)(-1+B-2)$$

$$16=(-2)(B-3) \Rightarrow B=-8+3 \Rightarrow \boxed{B=-5}$$

Therefore

$$|\underline{A}-\lambda\underline{I}|=(\lambda-3)(-\lambda^2-5\lambda-2)$$

The other eigenvalues are obtained from

$$-\lambda^2-5\lambda-2=0$$

$$\Delta=25-(4)(-1)(-2)=17$$

$$\lambda_{2,3}=\frac{5\mp\sqrt{17}}{-2} \Rightarrow \lambda_2=-\frac{1}{2}(5+\sqrt{17})$$

$$\lambda_3=-\frac{1}{2}(5-\sqrt{17})$$

Q4. (*25 pts*) Consider the following matrix $\boldsymbol{A}$.

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

(i) Find the eigenvalues.

(ii) Determine the eigenvectors.

(iii) Diagonalize $\boldsymbol{A}$.

(iv) Put the quadratic form $Q = 5\,x_1^2 + 4\,x_1x_2 + 5\,x_2^2 = 21$ into the simplified form of $Q = \alpha\, y_1^2 + \beta\, y_2^2 = 21$ through a proper transformation. What are the values of α and β?

(i) Eigenvalues

$$|\underline{A} - \lambda \underline{I}| = 0$$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \begin{array}{l} (5-\lambda)^2 - 2^2 = 0 \\ (5-\lambda-2)(5-\lambda+2) = 0 \end{array}$$

$$5-\lambda-2 = 0 \Rightarrow \lambda_1 = 3$$

$$5-\lambda+2 = 0 \Rightarrow \lambda_2 = 7$$

(ii) Eigenvectors : $(\underline{A} - \lambda \underline{I})\underline{x} = 0$

$$\underline{\lambda_1 = 3} \rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow 2x_1 + 2x_2 = 0\ ;\ x_2 = -1,\ x_1 = 1$$

$$\underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \text{Normalize } \underline{x}_1 \rightarrow \underline{x}_{1n} = \frac{\underline{x}_1}{\|\underline{x}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underline{x}_{1n} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\underline{\lambda_2 = 7} \rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow -2x_1 + 2x_2 = 0\ ;\ x_2 = 1,\ x_1 = 1$$

$$\underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \text{Normalize } \underline{x}_2 \rightarrow \underline{x}_{2n} = \frac{\underline{x}_2}{\|\underline{x}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{x}_{2n} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(iii) Form a matrix $\underline{X}$ using the normalized eigenvectors as

$$\underline{X} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \Rightarrow \underline{X}^{-1} = \frac{1}{\det(\underline{X})}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\det(\underline{X}) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 \Rightarrow \underline{X}^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

[redacted]

Note that $\underset{\sim}{X}^{-1} = \underset{\sim}{X}^{T}$ (Orthogonality) (1)

(If we had not normalize x_1 & x_2, this property would not be satisfied. This property is used in the quadratic form below)

Now calculate

$$\underset{\sim}{D} = \underset{\sim}{X}^{-1}\underset{\sim}{A}\underset{\sim}{X} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3/\sqrt{2} & 7/\sqrt{2} \\ -3/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

(iv) $Q = 5x_1^2 + 4x_1x_2 + 5x_2^2 = 21$

$$= \underbrace{[x_1 \;\; x_2]}_{x^T}\underbrace{\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = 21 \Rightarrow \underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{x} = 21 \qquad (2)$$

Since $\underset{\sim}{D} = \underset{\sim}{X}^{-1}\underset{\sim}{A}\underset{\sim}{X} \Rightarrow \underset{\sim}{A} = \underset{\sim}{X}\underset{\sim}{D}\underset{\sim}{X}^{-1}$

Using (1) $\rightarrow \underset{\sim}{A} = \underset{\sim}{X}\underset{\sim}{D}\underset{\sim}{X}^T$ (3)

Substituting (3) into (2)

$$\underset{\sim}{x}^T\underset{\sim}{X}\underset{\sim}{D}\underset{\sim}{X}^T\underset{\sim}{x} = 21 \qquad (4)$$

Subtitute $\underset{\sim}{y} = \underset{\sim}{X}^T\underset{\sim}{x}$ and $\underset{\sim}{y}^T = (\underset{\sim}{X}^T\underset{\sim}{x})^T = \underset{\sim}{x}\underset{\sim}{X}^T$, (4) becomes

$$\underset{\sim}{y}^T\underset{\sim}{D}\underset{\sim}{y} = 21 \Rightarrow [y_1 \;\; y_2]\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 21$$

$$3y_1^2 + 7y_2^2 = 21 \qquad \alpha = 3, \beta = 7$$

$$\left(\frac{y_1}{\sqrt{7}}\right)^2 + \left(\frac{y_2}{\sqrt{3}}\right)^2 = 1 \quad \text{An ellipse}$$

Check: $\underset{\sim}{y} = \underset{\sim}{X}^T\underset{\sim}{x} = \underset{\sim}{X}^{-1}\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = \underset{\sim}{X}\underset{\sim}{y} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (y_1+y_2)/\sqrt{2} \\ (-y_1+y_2)/\sqrt{2} \end{bmatrix}$

Substituting $x_1 = \frac{y_1+y_2}{\sqrt{2}}$, $x_2 = \frac{-y_1+y_2}{\sqrt{2}}$ into $5x_1^2 + 4x_1x_2 + 5x_2^2 = 21 \Rightarrow$

$$5\frac{(y_1+y_2)^2}{2} + 4\frac{y_2^2 - y_1^2}{2} + 5\frac{(-y_1+y_2)^2}{2} = 21 \qquad \frac{5-4+5}{2}y_1^2 + \frac{5+4+5}{2}y_2^2 = 21$$

$$3y_1^2 + 7y_2^2 = 21 \checkmark$$