Take Home Exam 1 - Solutions
Complex Numbers

Q1. Determine $z = \frac{3+i4}{4-i3}$.

$$= \frac{3+i4}{4-i3}\,\frac{(4+i3)}{(4+i3)} = \frac{1}{(16+9)}\left[(3+i4)(4+i3)\right] = \frac{1}{25}\left[(12+i^2 12)+i(9+16)\right]$$

$$= \frac{1}{25}\left[(\cancel{12}-\cancel{12})+i25)\right] = i$$

Q2. Determine $z = \sqrt{2+i2\sqrt{3}}$

$$2+i2\sqrt{3} = re^{i\theta} \Rightarrow r = \sqrt{4+12} = 4,\quad \theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$z = \sqrt{4\cdot e^{i(\frac{\pi}{3}+2\pi n)}} = 2e^{i(\frac{\pi}{6}+\pi n)} = 2\left[\left(\cos\left(\frac{\pi}{6}+\pi n\right)+i\sin\left(\frac{\pi}{6}+\pi n\right)\right)\right] \begin{cases} \sqrt{3}+i \\ -\sqrt{3}-i \end{cases}$$

Q3. Show that

(i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

$$\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2} = \overline{x_1+x_2+i(y_1+y_2)} = x_1+x_2-i(y_1+y_2)$$

$$= (x_1-iy_1)+(x_2-iy_2) = \bar{z}_1+\bar{z}_2$$

(ii) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

$$\overline{z_1 z_2} = \overline{(x_1+iy_1)(x_2+iy_2)} = \overline{x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)} = x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$$

$$= x_1(x_2-iy_2)-iy_1(x_2-iy_2) = (x_1-iy_2)(x_2-iy_2) = \bar{z}_1\cdot\bar{z}_2$$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{x_1+iy_1}{x_2+iy_2}\right)} = \overline{\left[\frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}\right]} = \overline{\left(\frac{x_1x_2+y_1y_2+i(-x_1y_2+x_2y_1)}{x_2^2+y_2^2}\right)}$$

$$= \frac{x_1x_2+y_1y_2+i(x_1y_2-x_2y_1)}{x_2^2+y_2^2} = \frac{(x_1-iy_1)(x_2+iy_2)}{x_2^2+y_2^2} = \frac{x_1-iy_1}{x_2-iy_2} = \frac{\bar{z}_1}{\bar{z}_2}$$

where $\bar{z}$ denotes the complex conjugate of the complex number z.

Q4. Determine $z = i^n$, n=0, 1, 2, 3,....

$n=0 \Rightarrow z = i^0 = 1$

$n=1 \Rightarrow z = i$

$n=2 \Rightarrow z = i^2 = -1$

$n=3 \Rightarrow z = i^3 = i^2 i = -i$

$n = i^4 = i^2\cdot i^2 = (-1)(-1) = 1$

$n = i^5 = i\cdot i^4 = i$

Then

$$z = i^n \begin{cases} 1 & n=0,4,8,\ldots \\ i & n=1,5,9,\ldots \\ -1 & n=2,6,10,\ldots \\ -i & n=3,7,11,\ldots \end{cases}$$

Q5. Find the value of z= $i^2+i^4+i^6+\cdots+i^{2n}$?

n is even:

$z = (-1)+(1)+(-1)\ldots+(1) = 0$

n is odd

$z = (-1)+(1)+(-1)\ldots+(-1) = -1$

Q6. Determine $z = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$.

$$e^{i\theta} = \cos\theta + i\sin\theta \rightarrow z = \frac{e^{i\theta}}{e^{-i\theta}} = e^{2\theta i}$$
$$= \cos 2\theta + i\sin 2\theta.$$

Q7. Find the value of $z = (i - i^2)^3$? $\rightarrow z = (i-(-1))^3 = (1+i)^3$

$$= [\sqrt{2}e^{i\pi/4}]^3 = \sqrt{2^3}\, e^{i3\pi/4} = 2\sqrt{2}(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4})$$
$$= 2\sqrt{2}\left(-\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}\right) = -2 + i2$$

Q8. If $z^3 = 1$, then determine $w = z^6 + z^7 + z^5$. ; $z^6 = (z^3)^2 = 1$

$$w = z^6 + z^7 + z^5 = \underbrace{z^6}_{1}(1 + z + z^{-1}) = 1 + z + z^{-1}$$
$$z^3 = 1 \Rightarrow z = \sqrt[3]{e^{i2\pi n}} = e^{i\frac{2\pi}{3}n} \Rightarrow w = 1 + e^{i\frac{2\pi}{3}n} + e^{-i\frac{2\pi}{3}n}$$
$$= 1 + 2\cos(\tfrac{2\pi}{3}n)$$

$n = 0 \Rightarrow w = 1 + 2 = 3$
$n = 1 \Rightarrow w = 1 + 2(-\tfrac{1}{2}) = 0$
$n = 2 \Rightarrow w = 1 + 2(-\tfrac{1}{2}) = 0$

Q9. If $(1+i)(x+iy) = 2+i4$ then determine x.

$$(x-y) + i(x+y) = 2 + i4$$
$$x - y = 2$$
$$+\ x + y = 4$$
$$2x = 6 \Rightarrow x = 3 \Rightarrow y = 1 \Rightarrow z = 3 + i$$

Q10. Determine z if $w = \frac{z-1}{z+1}$ is purely imaginary.

$$z = x + iy \Rightarrow w = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2+y^2}$$
$$w = \frac{[(x^2-1)+y^2] + i[y(x+1) - y(x-1)]}{(x+1)^2+y^2} = \frac{(x-1)^2 + y^2 + i2y}{(x+1)^2+y^2}$$

w is purely imaginary $\Rightarrow Re(w) = 0 \Rightarrow (x-1)^2 + y^2 = 0$ (Both are square!)
$\Rightarrow (x-1)^2 = 0$ & $y^2 = 0 \Rightarrow x = 1$ & $y = 0$

$z = 1 \Rightarrow w = 0$

(Bonus). Using Taylor series expansion show that $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \Rightarrow e^{i\theta} = 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \dots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)}_{\sin\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$