HOMEWORK 10 – ANSWER KEY – EEE 281

Q1 : $y'' + 6y' + 8y = 0$

$$\left.\begin{array}{l} u = y \quad v = y' \quad u' = y' = v \\ v' = y'' = -6y' - 8y = -6v - 8u \end{array}\right\} \left.\begin{array}{l} ① \ u' = 0u + v \\ ② \ v' = -8u - 6v \end{array}\right\} \begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\left.\begin{array}{l} \delta = |A - \lambda I| \\ |A - \lambda I| = \begin{bmatrix} -\lambda & 1 \\ -8 & -6-\lambda \end{bmatrix} \end{array}\right\} \begin{array}{l} (-\lambda)(-6-\lambda) + 8 = 0 \\ \lambda^2 + 6\lambda + 8 = 0 \\ \lambda_1 = -2 \quad \lambda_2 = -4 \end{array}$$

For $\lambda_1 = -2$

$$\begin{bmatrix} 2 & 1 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 2x + y = 0 \\ -8x + 4y = 0 \\ x = 1 \quad y = -2 \end{array}\right\} v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

for $\lambda_2 = -4$

$$\begin{bmatrix} 4 & 1 \\ -8 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 4x + y = 0 \\ -8x - 2y = 0 \\ x = 1 \quad y = -4 \end{array}\right\} \begin{bmatrix} x \\ y \end{bmatrix} = v_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 . e^{\lambda_1 t} . v_1 + c_2 . e^{\lambda_2 t} . v_2$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{-4t} \\ -2c_1 e^{-2t} - 4c_2 e^{-4t} \end{bmatrix} \Rightarrow \begin{array}{l} u = c_1 . e^{-2t} + c_2 . e^{-4t} \\ y = u = c_1 e^{-2t} + c_2 e^{-4t} \end{array}$$

Q2: $y'' + 2y' + 2y = 0$

$y = x_1 \qquad y' = x_2$

$x_1' = y' = x_2$

$x_2' = y'' = -2y - 2y' = -2x_1 - 2x_2$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0$$

$(\lambda+1)^2 + 1 = 0 \rightarrow \lambda = -1 \pm i$

for $-1 + i = \lambda$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-t} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right\} + c_2 \cdot e^{-t} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t \right\}$$

$c_1 = 1 \qquad c_2 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos t + \sin t) \\ -2e^{-t} \cdot \sin t \end{bmatrix} \qquad y(t) = e^{-t}(\cos t + \sin t)$$

Q3: $$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3y_1 + y_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$(3-\lambda)(1-\lambda) - 1 = 0 \qquad 3 - 3\lambda - \lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 2 = 0 \qquad \lambda = 2 \pm \sqrt{2}$

$$\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 . e^{\lambda_1 . x} . v_1 + c_2 e^{\lambda_2 . x} . v_2$$

Q4: $y_1' = y_2 \qquad y_2' = -y_1 + y_3 \qquad y_3' = -y_2$

$$y' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1+y_3 \\ -y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Q5: $\left.\begin{aligned} y_1' &= 2y_1 + 2y_2 \\ y_2' &= 3y_1 - 2y_2 \end{aligned}\right\} \quad \begin{aligned} y_1(0) &= 12 \\ y_2(0) &= 4 \end{aligned}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightarrow A = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix}$$

$$S = A - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} \rightarrow \det S = -\overbrace{(2-\lambda)}^{\lambda-2}(2+\lambda) - 6 = 0$$

$$\det S = \lambda^2 - 4 - 6 = 0 \quad \lambda_{1,2} = \sqrt{10}, -\sqrt{10}$$

for $\lambda_1 = \sqrt{10} \rightarrow v_1 = \begin{bmatrix} 2/3 + \sqrt{10}/3 \\ 1 \end{bmatrix}$

for $\lambda_2 = -\sqrt{10} \rightarrow v_2 = \begin{bmatrix} 2/3 - \sqrt{10}/3 \\ 1 \end{bmatrix}$

$$Y(t) = c_1 . v_1 . e^{\lambda_1 t} + c_2 . v_2 . e^{\lambda_2 t}$$

Apply initial conditions.

$$y_1(t) = c_1\left(2/3 + \frac{\sqrt{10}}{3}\right) e^{\sqrt{10}t} + c_2\left(2/3 - \sqrt{10}/3\right) e^{-\sqrt{10}t}$$

$$y_2(t) = c_1 . e^{\sqrt{10}t} + c_2 e^{-\sqrt{10}t}$$

$$c_1 = \frac{7\sqrt{10}}{5} + 2 \qquad c_2 = 2 - 7\frac{\sqrt{10}}{5}$$

$$y_1(t) = \left(\frac{7\sqrt{10}}{5} + 2\right)\left(2/3 + \frac{\sqrt{10}}{3}\right) e^{\sqrt{10}t} + \left(2 - \frac{7\sqrt{10}}{5}\right)\left(2/3 - \frac{\sqrt{10}}{3}\right) e^{-\sqrt{10}t}$$

$$y_2(t) = \left(\frac{7\sqrt{10}}{5} + 2\right) e^{\sqrt{10}t} + \left(2 - \frac{7\sqrt{10}}{5}\right) e^{-\sqrt{10}t}$$