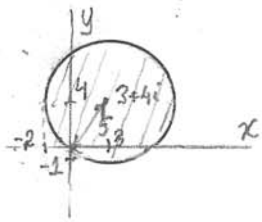
EEE 281 Engineering Mathematics I
Take-Home Exam Part 2
Due: 24 Oct 2023

Take Home Exam 2
Complex Functions

Q1. Determine and sketch or graph the sets in the complex plane given by

(i) $|z-3-4i| \le 5 \Rightarrow |z-(3+4i)| \le 5$

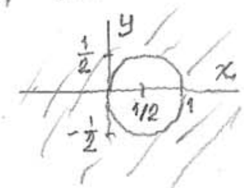
Inside the circle centered at $3+4i$ with radius 5.



(ii) $\mathrm{Re}(1/z) \le 1$

$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \Rightarrow \mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2} \le 1 \Rightarrow x \le x^2+y^2 \Rightarrow 0 \le x^2-x+y^2$

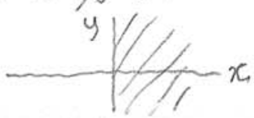
$\Rightarrow 0 \le (x-\frac{1}{2})^2+y^2-\frac{1}{4} \Rightarrow \frac{1}{4} \le (x-\frac{1}{2})^2+y^2 \Rightarrow$ Outside circle centered at $(\frac{1}{2},0)$ with radius $\sqrt{\frac{1}{4}} = 1/2$



(iii) $|z+1| \ge |z-1|$

$|x+iy+1| \ge |x+iy-1| \Rightarrow \sqrt{(x+1)^2+y^2} \ge \sqrt{(x-1)^2+y^2}$

$(x+1)^2+y^2 \ge (x-1)^2+y^2 \Rightarrow (x+1)^2-(x-1)^2 \ge 0 \Rightarrow 4x \ge 0$ or $x \ge 0$



Right half plane where $x \ge 0$.

Q2. Find the value of the derivative of the following functions at the specified points.

(i) $f(z) = (3z^2+iz)^3$ at $z=0$. $f(z)$ is continuous at z.

$f'(z) = 3(3z^2+iz)^2(6z+i)$

$f'(0) = 0$

(ii) $f(z) = \frac{z-1}{z+i}$ at $z=1$. $\Rightarrow f(z)$ is continuous at $z=1$ (singularity is at $(0,-i)$

$f'(z) = \frac{(1)(z+i)-(z-1)}{(z+i)^2} \Rightarrow f'(1) = \frac{1+i}{(1+i)^2} = \frac{1}{1+i} = \frac{1-i}{2} = \frac{1}{2}-\frac{1}{2}i$

Q3. Are the following functions analytic? (Using proper form of Cauchy-Riemann Equations)

(i) $f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2-y^2+2xyi}{x^2+y^2}$

$u(x,y) = \frac{x^2-y^2}{x^2+y^2}$, $v(x,y) = \frac{2xy}{x^2+y^2}$

$u_x = \frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$; $u_y = \frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$

$v_x = \frac{2y(x^2+y^2)-2x(2xy)}{(x^2+y^2)^2} = \frac{2y(-x^2+y^2)}{(x_2+y_2)^2}$, $v_y = \frac{2x(x^2+y^2)-2y(2xy)}{(x^2+y^2)^2} = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$

$u_x \ne v_y$ & $u_y \ne -v_x \Rightarrow f(z)$ is not analytic

(ii) $f(z) = Re(z^2) + i\,Im(z^2)$ $\quad z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$f(z) = \underbrace{x^2-y^2}_{u} + i\underbrace{2xy}_{v} \rightarrow u(x,y) = x^2 - y^2 \Rightarrow u_x = 2x \;;\; u_y = -2y$
$\qquad v(x,y) = 2xy \Rightarrow v_x = 2y \;;\; v_y = 2x$ $\quad \left.\right\} \begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$
So $f(z)$ is analytic.

(iii) $f(z) = \ln r + i\theta$ where $z = r\,e^{i\theta}$ $\rightarrow u(r,\theta) = \ln r$ & $v(r,\theta) = \theta$

$u_r = \frac{1}{r} v_\theta$ & $v_r = -\frac{1}{r} u_\theta$ must be satisfied. (Because of the unit compatibily, derivates with respect to θ must be divided by 'r'.

$u_r = \frac{1}{r}$ & $u_\theta = 0$ $\{ v_r = 0,\ v_\theta = 1 \Rightarrow u_r = \frac{1}{r} v_\theta$ ✓ & $v_r = \frac{1}{r} u_\theta$ ✓ $\rightarrow$ So f is analytic

Q4. Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x,y) + i\,v(x,y)$

(i) $u(x,y) = x^3 + 3xy^2$ $\Rightarrow u_x = 3x^2 + 3y^2 \Rightarrow u_{xx} = 6x$
$\qquad u_y = 6xy \Rightarrow u_{yy} = 6x$ $\quad \left.\right\}$ $u_{xx} + u_{yy} = 12x \neq 0$ Not harmonic!

Note that $u(x,y) = -x^3 + 3xy^2$ would be harmonic!

(ii) $v(x,y) = e^x cosy$ $\Rightarrow v_x = e^x \cos y \Rightarrow v_{xx} = e^x \cos y$
$\qquad v_y = -e^x \sin y \Rightarrow v_{yy} = -e^x \cos y$ $\quad \left.\right\}$ $v_{xx} + v_{yy} = 0$ so $v(x,y)$ is harmonic.

$u_x = v_y = -e^x \sin y \Rightarrow u(x,y) = \int -e^x \sin y\, \partial x = -e^x \sin y + h(y)$ $\{ u_y = -e^x \cos y + h'(y)$

$u_y = -v_x \Rightarrow -e^x \cos y + h'(y) = -e^x \cos y \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow u = -e^x \sin y + C$

$f(z) = u(x,y) + i\,v(x,y) = (-e^x \sin y + c) + i e^x \cos y$

Q5. Determine a and b so that the given function is harmonic and find a harmonic conjugate.

(i) $u(x,y) = ax^3 + bxy$ $\Rightarrow u_x = 3ax^2 + 3by \Rightarrow u_{xx} = 6ax$
$\qquad u_y = bx \Rightarrow u_{yy} = 0$ $\quad \left.\right\}$ $u_{xx} + u_{yy} = 0$; $6ax + 0 = 0 \Rightarrow a = 0$

The $\underline{u(x,y) = bxy}$ $\Rightarrow u_x = by$ & $u_y = bx$

$u_x = v_y \Rightarrow by = v_y \Rightarrow v = \int v_y\, \partial y = \int by\, \partial y = \frac{1}{2} by^2 + h(x) \Rightarrow v_x = h'(x)$

$u_y = -v_x \Rightarrow bx = -h'(x) \Rightarrow h(x) = \int -bx\, dx = -\frac{b}{2} x^2 + C \Rightarrow v(x,y) = -\frac{b}{2} x^2 + \frac{b}{2} y^2 + C$

$\Rightarrow f(z) = bxy + i\left(-\frac{b}{2} x^2 + \frac{b}{2} y^2 + C\right)$ $\}$ Note that $u_x = v_y$ & $u_y = -v_x$ are satisfied.

(ii) $v(x,y) = \cosh ax \cos by$ $\Rightarrow v_x = a \sinh ax \cos by \Rightarrow v_{xx} = a^2 \cosh ax \cos by$
$\qquad v_y = -b \cosh ax \sin by \Rightarrow v_{yy} = -b^2 \cosh ax \cos by$ $\quad \left.\right\}$ $v_{xx} + v_{yy} = 0$; $a^2 = b^2 \Rightarrow a = \mp b$

Let $a = b \Rightarrow v(x,y) = \cosh ax \cos ay$

$u_x = v_y = -a \cosh ax \sin ay \Rightarrow u(x,y) = -\int a \cosh ax \sin ay\, \partial x = -\sinh ax \sin ay + h(y)$

$u_y = -v_x \Rightarrow -a \sinh ax \cos ay + h'(y) = -a \sinh ax \cos ay \Rightarrow h'(y) = 0 \Rightarrow h(y) = C$

$\Rightarrow f(z) = (-\sinh ax \sin ay + c) + i \cosh ax \cos ay$ $\}$ Note that $u_x = v_y$ & $u_y = -v_x$ are satisfied.

Q-bonus. Determine z such that $\sin z = 5$. $\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 5 \Rightarrow e^{iz} - e^{-iz} = 10i$

Let $e^{iz} = t \Rightarrow t - \frac{1}{t} = 10i \Rightarrow t^2 - 10it - 1 = 0$ (Quadratic equation with $a=1, b=10i, c=-1$

$t_{1,2} = \frac{10i \mp \sqrt{-100+4}}{2} = (5 \mp \sqrt{24})i < \begin{matrix} 9.899i \\ 0.101i\ (=1/t_1) \end{matrix}$

$e^{iz} = e^{i(x+iy)} = e^{-y} e^{ix} = e^{-y}(\cos x + i \sin x) = 9.899i \Rightarrow \underbrace{\cos x = 0 \ \&\ \sin x = 1}_{x = \frac{\pi}{2} + 2\pi n}$ & $e^{-y} = 9.899$

$y = -\ln 9.899 = -2.29$ ($y = +2.29$ for $t_2 = 0.101$) Then $z = (\frac{\pi}{2} + 2\pi n) \mp 2.29$