Take Home Exam 3
Line Integral of Complex Functions

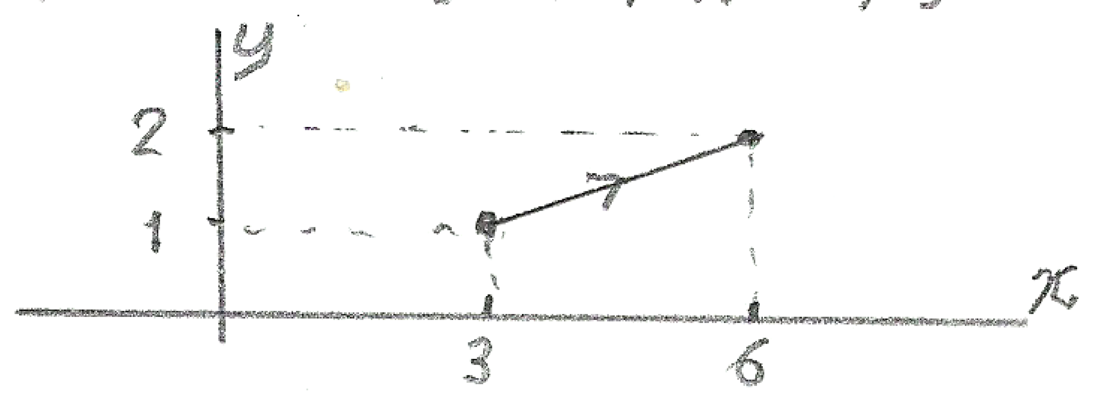
Q1. Find and sketch the following paths.

(i) $(3+i)t \quad (1 \le t \le 2)$

$Z = 3t + it \,;\, x = 3t,\ y = t \Rightarrow \frac{y}{x} = \frac{1}{3}$

$y = \frac{1}{3}x$ $\quad$ ($t = 1 \Rightarrow x = 3,\ y = 1$

(Linear) $\quad t = 2 \rightarrow x = 6,\ y = 2$



(ii) $2 + i + (1-i)t \quad (0 \le t \le 2)$

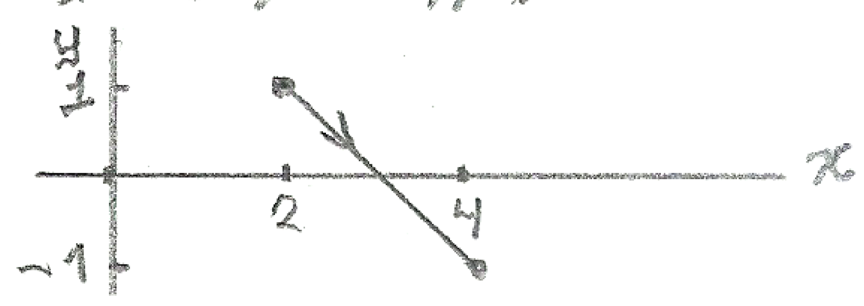
$Z = 2 + i + t - ti = 2 + t + (1-t)i$

$x = 2 + t,\ y = 1 - t$

$t = x - 2 \Rightarrow y = 1 - (x-2) = -x + 1$ (Linear)

$t = 0 \Rightarrow x = 2,\ y = 1$

$t = 2 \Rightarrow x = 4,\ y = -1$



(iii) $1 + i + 4e^{it} \quad (0 \le t \le 2\pi)$

$Z = 1 + i + 4(\cos t + i \sin t)$

$= (1 + 4\cos t) + (1 + 4\sin t)\,i \Rightarrow$

$x = 1 + 4\cos t \Rightarrow \frac{x-1}{4} = \cos t$ | Use a known relationship

$y = 1 + 4\sin t \Rightarrow \frac{y-1}{4} = \sin t$

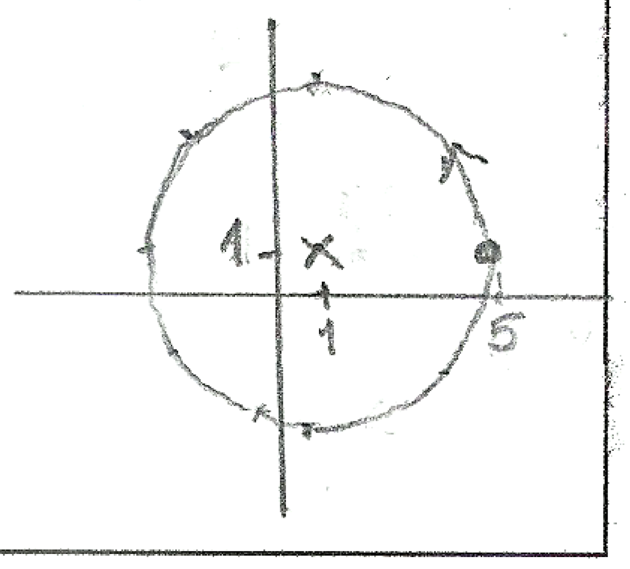
$\cos^2 t + \sin^2 t = 1 \Rightarrow \left(\frac{x-1}{4}\right)^2 + \left(\frac{y-1}{4}\right)^2 = 1$

$(x-1)^2 + (y-1)^2 = 4^2$

This describes a circle centered at (1,1) with radius 4.

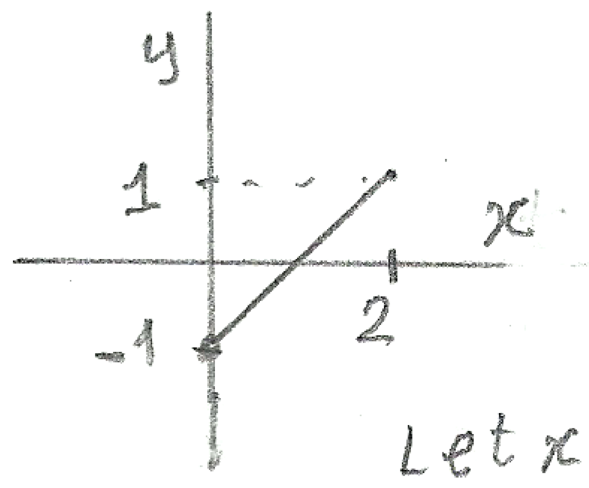
$t = 0 \Rightarrow x = 5,\ y = 1$

$t = 2\pi \Rightarrow x = 5,\ y = 1$



Q2. Find a parametric representation of the following paths and sketch them.

(i) Line segment from (0,-1) to (2,1).



$y = mx + b$

$x = 0 \Rightarrow y = b = -1$

$y = mx - 1$

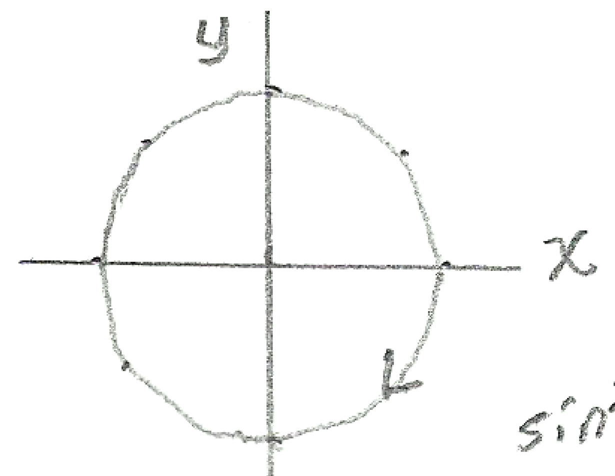
$(2,1) \Rightarrow 1 = m \cdot 2 - 1 \Rightarrow m = 1$

$y = 2x - 1$

Let $x = t \Rightarrow y = 2t - 1$

$z = t + i(2t-1) \quad 0 \le t \le 2$

(ii) Unit circle, clockwise



$x^2 + y^2 = 1$

$z = re^{i\theta} \quad (r = 1)$

$= \underbrace{\cos\theta}_{x} + i\underbrace{\sin\theta}_{y}$

$\sin^2\theta + \cos^2\theta = 1$

$z = \cos\theta + i\sin\theta \quad -2\pi \le \theta \le 0$ (Clockwise!)

(i) Ellipse $x^2 + 4y^2 = 16$, counterclockwise (Try to use $\cos^2 t + \sin^2 t = 1$!)

$\frac{x^2}{16} + \frac{y^2}{4} = 0$

$\frac{x^2}{16} = \cos^2 t \Rightarrow x = 4\cos t$

$\frac{y^2}{4} = \sin^2 t \Rightarrow y = 2\sin t$

$z = \underbrace{4\cos t}_{x} + i\,\underbrace{2\sin t}_{y} \quad 0 \le t \le 2\pi$

Q3. Calculate the following integral

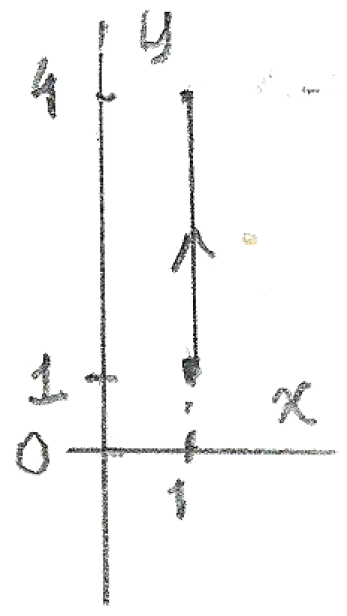
$$\int_C Re\, z\, dz = \int_C Re(x+iy)\,dz = \int_C x\,dz$$

over the paths defined below.

(i) C: The shortest path from $1+i$ to $1+4i$.

(ii) C: The parabola $y = x^2$ from $1+i$ to $1+4i$.

(i)



Over the path $x=1,\ y=t$

$z = x+iy$

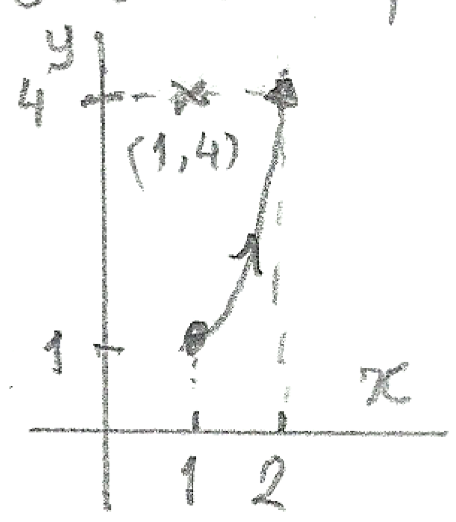
$= 1+it \quad 1 \le t \le 4$

$dz = i\,dt$

$$\int_C x\,dz = \int_1^4 \overset{Re(z)}{(1)}\,\overset{dz}{i\,dt} = it\Big|_1^4$$

$= 3i$

(ii) Over the path $x=t,\ y=x^2=t^2$



The point $1+4i$ is not on $y=x^2$. Not proper

Solve for the points from $\underline{1+4i}$ to $\underline{2+4i}$

Over the curve $z = x+iy$

$x=t,\ y=t^2 \rightarrow z = t+it^2$

$dz = (1+i2t)dt \quad 1\le t\le 2$

$$\int_C Re(z)\,dz = \int_C x\,dz = \int_C \underbrace{t}_{Re(z)}\underbrace{(1+i2t)\,dt}_{dz} = \int_1^2 t(1+i2t)dt = \int_1^2 (t+i2t^2)dt = \frac{t^2}{2}+\frac{2}{3}it^3\Big|_1^2$$

$$= \frac{3}{2}+\frac{14}{3}i$$

Q4. Express $f(z)$ in terms of partial fractions and integrate it over the circle $|z| = 2$ (counterclockwise) where

(i) $f(z) = \frac{2z+1+i}{z^2+1}$

(ii) $f(z) = \frac{z+i}{z^2+4z}$

(iii) $f(z) = \frac{4z+1}{z^2+9}$

(i) $$f(z) = \frac{2z+1+i}{z^2+1} = \frac{2z+1+i}{(z+i)(z-i)} = \frac{A}{z-i}+\frac{B}{z+i} = \frac{(A+B)z+(A-B)i}{(z-i)(z+i)}$$

$$\left.\begin{aligned}&A+B=2\\&(A-B)i = 1+i \rightarrow A-B = \frac{1+i}{i} = -i+1 = 1-i\end{aligned}\right\}\quad \left.\begin{aligned}&A+B=2\\&A-B=1-i\end{aligned}\right\}\quad \begin{aligned}&2A = 3-i\\&A=\frac{3}{2}-\frac{1}{2}i\end{aligned}$$

$$A-B = 1-i \Rightarrow B = A-1+i = \frac{3}{2}-\frac{1}{2}i-1+i = \frac{1}{2}+\frac{1}{2}i$$

$$A = \frac{3}{2}-\frac{1}{2}i\ ;\ B = \frac{1}{2}+\frac{1}{2}i$$

$$f(z) = \frac{2z+1+i}{z^2+1} = \frac{1}{2}\left(\frac{3-i}{z-i}\right)+\frac{1}{2}\left(\frac{1+i}{z+i}\right)$$

$$\oint_C f(z)\,dz = \int_C \frac{\overbrace{\frac{1}{2}\,3-i}^{g_1(z)}}{z-i}dz + \int_C \frac{\overbrace{\frac{1}{2}\,1+i}^{g_2(z)}}{z+i}dz \qquad (z=\mp i \text{ are within } C.)$$

$$= 2\pi i\, g_1(z=i) + 2\pi i\, g_2(z=-i)$$

where $g_1(z) = \frac{1}{2}(3-i) \Rightarrow g_1(z=i) = \frac{1}{2}(3-i)$

$g_2(z) = \frac{1}{2}(1+i) \Rightarrow g_2(z=-i) = \frac{1}{2}(1+i)$

$$\oint_C f(z)\,dz = (2\pi i)\left(\tfrac{1}{2}\right)(3-i)+(2\pi i)\left(\tfrac{1}{2}\right)(1+i) = \frac{2\pi i}{2}(4) = 4\pi i$$

(ii) $f(z) = \dfrac{z+i}{z^2+4z} = \dfrac{z+i}{z(z+4)} = \dfrac{A}{z} + \dfrac{B}{z+4} = \dfrac{(A+B)z + 4A}{z(z+4)}$

$A+B=1$

$4A = i \rightarrow A = \frac{1}{4}i \Rightarrow B = 1-A = 1-\frac{1}{4}i$

$f(z) = \dfrac{z+i}{z^2+4} = \dfrac{i/4}{z} + \dfrac{1-\frac{1}{4}i}{z+4}$

$$\oint_C f(z)\,dz = \oint_C \frac{\overbrace{i/4}^{g(z)}}{\underbrace{z}_{z_0=0}}\,dz + \underbrace{\oint_C \frac{1-i/4}{z+4}\,dz}_{=0} = 2\pi i\, g(z_0) = 2\pi i\left(\frac{i}{4}\right) = -\frac{\pi}{2}$$

(since $z=-4$ is outside C)

or

$$\oint_C \frac{z+i}{z^2+4z}\,dz = \oint_C \frac{z+i}{z(z+4)}\,dz = \oint_C \frac{\overbrace{(z+i)/(z+4)}^{g(z)}}{\underbrace{z}_{z_0=0}}\,dz = 2\pi i\, g(z_0)$$

$$= 2\pi i \left.\frac{z+i}{z+4}\right|_{z=0} = 2\pi i\left(\frac{i}{4}\right) = -\frac{\pi}{2}$$

(iii) $f(z) = \dfrac{4z+1}{z^2+9} = \dfrac{4z+1}{(z-3i)(z+3i)} = \dfrac{A}{z-3i} + \dfrac{B}{z+3i} = \dfrac{(A+B)z+(A-B)3i}{(z-3i)(z+3i)}$

$A+B=4$

$(A-B)3i = 1 \Rightarrow A-B = \frac{1}{3i} = -\frac{1}{3}i$

$\Big\}$ $2A = 4-3i \Rightarrow A = 2-\frac{3}{2}i$

$2B = 4+3i \Rightarrow B = 2+\frac{3}{2}i$

$$\oint_C f(z)\,dz = \underbrace{\oint_C \frac{2-\frac{3}{2}i}{z-3i}\,dz}_{=0} + \underbrace{\oint_C \frac{2+\frac{3}{2}i}{z+3i}\,dz}_{=0} = 0$$

(since $z_0 = 3i$ is outside C) (since $z_0=-3i$ is outside C)

or

$\oint_C f(z)\,dz = 0$ since both $z_0 = \mp 3i$ are outside C.

Q5. Calculate the following integral

$$\oint_C \left(\frac{2z+4-i}{(z-i)^2}\right) dz$$

over the closed path C defined as C: The circle $|z-i| = 1$, clockwise

$$\oint_C \frac{2z+4-i}{(z-i)^2}\, dz = -2\pi i\, f'(z_0 = i) \quad \text{where } f(z) = 2z+4-i \quad \left(\begin{array}{l}\text{clock wise} \\ \text{brings } (-1) \text{ factor}\end{array}\right)$$

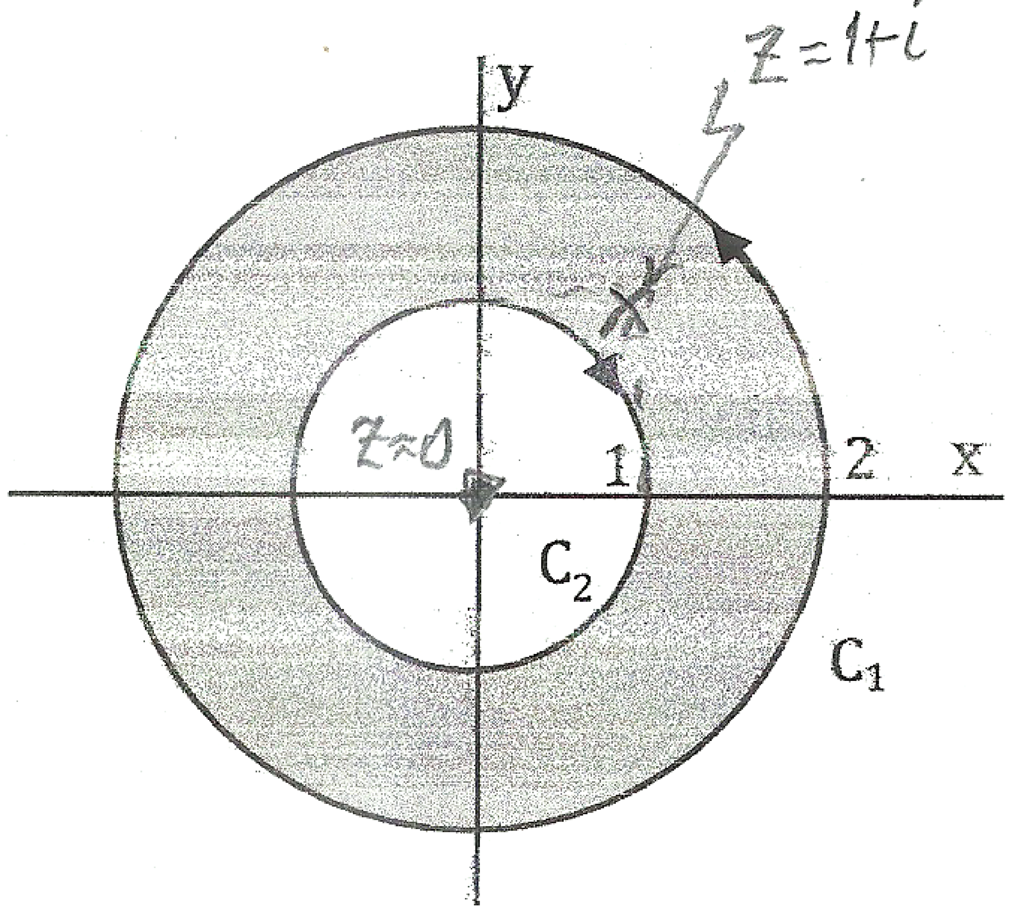
$$f'(z) = 2 \Rightarrow f'(z_0) = 2, \text{ then}$$

$$\oint_C \frac{2z+4-i}{(z-i)^2}\, dz = -(2\pi i)(2) = -4\pi i$$

Bonus. Integrate

$$\oint_C \frac{e^z}{z\,(z-1-i)} dz$$

where the contour C consists of the circle $|z| = 2$ (C_1) counterclockwise and the circle $|z| = 1$ (C_2) clockwise. (C_1 and C_1 constitute the boundary for the ring-shaped domain)



z=0 is outside the region described by C, whereas z = 1+i is within the region.

Then

$$\oint_C \frac{e^z}{z(z-1-i)} dz = \oint_C \frac{\overbrace{e^z/z}^{f(z)}}{\underbrace{z-1-i}_{\to z_0 = 1+i}} dz = \oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

where $f(z) = \frac{e^z}{z}$ and $z_0 = 1+i$

$$\oint_C \frac{e^z}{z(z-1-i)} = \frac{e^z}{z}\Big|_{z_0 = 1+i} = \frac{e^{1+i}}{1+i} = \frac{e^1 \cdot e^i}{\sqrt{2}\, e^{i\pi/4}} = \frac{e}{\sqrt{2}} e^{(1-\frac{\pi}{4})i}$$

$$= \frac{e}{\sqrt{2}}\left[\cos\left(1-\frac{\pi}{4}\right) + i\sin\left(1-\frac{\pi}{4}\right)\right]$$