HOMEWORK 6 – ANSWER KEY – EEE 281

Q1: (i) $\begin{bmatrix} 4 & -6 \\ 3 & 8 \end{bmatrix} \rightarrow S = \begin{bmatrix} 4-\lambda & -6 \\ 3 & 8-\lambda \end{bmatrix}$

$\det S = \lambda^2 - 12\lambda + 50 = 0 \rightarrow \lambda_{1,2} = \pm i\sqrt{14} + 6$

for $-i\sqrt{14}+6$ | for $i\sqrt{14}+6$

$v_1 = \begin{bmatrix} \frac{-i\sqrt{14}-2}{3} \\ 1 \end{bmatrix}$ $\qquad v_2 = \begin{bmatrix} \frac{i\sqrt{14}-2}{3} \\ 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \rightarrow S = \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix}$

$\lambda^2 - 9 = 0 \rightarrow \lambda_1 = 3 \quad \lambda_2 = -3$

for $\lambda_1 = 3$

$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = x_2 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = -3$

$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 12 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 12 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$(-\lambda+3)(-\lambda+4)(-\lambda+3) + 5.12.0 + 3.0.0 - 0 - 0 - (-\lambda+3).0.5$

$-\lambda^3 + 10\lambda^2 - 33\lambda + 36 \rightarrow \lambda_1 = 3 \qquad \lambda_2 = 3 \qquad \lambda_3 = 4$

① $\lambda_3 = 4$

$\left[\begin{array}{ccc|c} -1 & 5 & 3 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 + 3R_2 \rightarrow R_1$

$x_3 = 0$

$x_1 = 5x_2$

$v_3 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda_{1,2}=3$

$$\left[\begin{array}{ccc|c} 0 & 5 & 3 & 0 \\ 0 & 1 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3=0$

$x_2=0$

$x_1=x_1$

$$v_{1,2}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(iv) $$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \rightarrow \det A = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix}$$

$$-\lambda^3+7\lambda^2-10\lambda+4=0$$

① $\lambda_1=1$

② $\lambda_2=-\sqrt{5}+3$

③ $\lambda_3=+\sqrt{5}+3$

① for λ_1

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad v_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

② for $\lambda_2=-\sqrt{5}+3$

$$\left[\begin{array}{ccc|c} \sqrt{5}-1 & 0 & 4 & 0 \\ 0 & \sqrt{5}-2 & 0 & 0 \\ 1 & 0 & \sqrt{5}+1 & 0 \end{array}\right] \dashrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \sqrt{5}+1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad v_2=\begin{bmatrix} -\sqrt{5}-1 \\ 0 \\ 1 \end{bmatrix}$$

③ for $\lambda_3=\sqrt{5}+3$

$$\left[\begin{array}{ccc|c} -\sqrt{5}-1 & 0 & 4 & 0 \\ 0 & -\sqrt{5}-2 & 0 & 0 \\ 1 & 0 & -\sqrt{5}+1 & 0 \end{array}\right] \dashrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\sqrt{5}+1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad v_3=\begin{bmatrix} +\sqrt{5}-1 \\ 0 \\ 1 \end{bmatrix}$$

Q2: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}\cdot a_{21} = 0$$

$\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \Rightarrow$ Characteristic Equation

There are 2 roots; λ_1 and λ_2

Sum of the roots $= a_{11} + a_{22}$

↳ $ax^2 + \underline{b}x + \underline{c}$

$\frac{-b}{a} = \text{sum}$ product $= \frac{c}{a}$

Q3: (i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = A \rightarrow A^T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \rightarrow$ No Symmetric, No Skew-symmetric

$A \cdot A^T = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2-b^2 & 2ab \\ -2ab & -b^2+a^2 \end{bmatrix} \rightarrow$ Not orthoporal

(ii) $B = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \rightarrow B^T = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

↳ Not symmetric
↳ Skew-symmetric

! $A^T = A$: symmetric
$A^T = -A$: skew-symm.
$A^T \cdot A = I$: orthoponal

$\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \rightarrow$ Not orthoponal

(iii) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \rightarrow C^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix}$

↳ Not Symmetric
↳ Not Skew-symmetric

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} = \text{---} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↳ orthogonal

(iv) $\begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \\ -\frac{4}{9} & -\frac{7}{9} & \frac{4}{9} \\ \frac{8}{9} & -\frac{4}{9} & \frac{1}{9} \end{bmatrix} = D \qquad D^T = \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 4/9 & -7/9 & -4/9 \\ 8/9 & 4/9 & 1/9 \end{bmatrix}$

Not symmetric
Not skew-symmetric

$D^T . D = \dots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ orthogonal

<u>Q4</u>: $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

a) $P^{-1} . A . P \qquad \det(P) = 4 - 6 = -2$

$P^{-1} = \begin{bmatrix} 0.5 & 1 \\ 1.5 & 2 \end{bmatrix}$

$\hat{A} = \begin{bmatrix} 0.5 & 1 \\ 1.5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 1.5 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -25 & 11 \end{bmatrix} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$

b) $A - \lambda I \Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = -(3-\lambda)(\lambda+3) - 16 = 0$

$\lambda^2 = 9 + 16 = 25 \qquad \lambda_{1,2} = 5 \pm$

$\hat{A} - \lambda I \Rightarrow \begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = (\lambda - 25)(25 + \lambda) + 600 = 0$

$\lambda^2 - 625 + 600 = 0$

$\lambda^2 = 25 \rightarrow \lambda_{1,2} = \pm 5$

Same eigenvalues

c) for A $\lambda_1=5 \rightarrow -2x_1+4x_2=0$

$$\begin{bmatrix}-2 & 4\\ 4 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-2 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$v_1=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$P^{-1}.x=\begin{bmatrix}0.5 & 1\\ 1.5 & 2\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 5\end{bmatrix}$$

$\hat{A}$ $\lambda_1=5$

$$\begin{bmatrix}-30 & 12\\ -50 & 20\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-30 & 12 & 0\\ -50 & 20 & 0\end{array}\right]=\left[\begin{array}{cc|c}-30 & 12 & 0\\ 0 & 0 & 0\end{array}\right]$$

$-5 \quad 2$
$30 \quad -12$

$-30x_1+12x_2=0 \qquad \hat{v}_1=\begin{bmatrix}2\\ 5\end{bmatrix}$

Q5: (i) $A=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}$

* from above question: $\lambda_{1,2}=\pm 5$

for $\lambda=5 \rightarrow v_1=\begin{bmatrix}2\\ 1\end{bmatrix}$

for $\lambda=-5 \rightarrow v_2=\begin{bmatrix}-0.5\\ 1\end{bmatrix}$

Diaponalisation: PDP^{-1}

$P=\begin{bmatrix}-0.5 & 2\\ 1 & 1\end{bmatrix} \quad D=\begin{bmatrix}-5 & 0\\ 0 & 5\end{bmatrix}$

$\det P=-2.5$

$P^{-1}=\begin{bmatrix}-0.4 & 0.8\\ 0.4 & 0.2\end{bmatrix}$

(ii) $B=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$

$\begin{bmatrix}1-\lambda & 0\\ 2 & -1-\lambda\end{bmatrix} \rightarrow \lambda^2-1=0$

① $\lambda_1=1$
$\lambda_2=-1$

for $\lambda_1=1$

$$\begin{bmatrix}0 & 0\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$v_1=\begin{bmatrix}1\\ 1\end{bmatrix}$$

for $\lambda_2=-1$

$$\begin{bmatrix}2 & 0\\ 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$2x_1$

$$V_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$P=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \quad D=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} \quad P^{-1}=\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}$$

Q6: (i) $Q = x_1^2 + 6 . x_1 x_2 + 9x_2^2 = 10$

Quadratic form $\Rightarrow Q = x^T . A . x$

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \longrightarrow (S = A - I\lambda) \rightarrow S = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 9-\lambda \end{bmatrix}$$

$$\det(S) = (1-\lambda)(9-\lambda) - 9 = 0 = \lambda^2 - 10\lambda \rightarrow \lambda = 0 \text{ and } \lambda = 10$$

for $\lambda = 0$

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 = 0$

$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

for $\lambda = 10$

$$\begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 + x_2 = 0$

$$v_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$Q = \underbrace{0 . y_1^2}_{0} + 10 . y_2^2 = 10 \qquad y_2 = \pm 1 \qquad \Big\} \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$x = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow x_1 = \frac{-3y_1 + y_2}{\sqrt{10}} \qquad x_2 = \frac{y_1 + 3y_2}{\sqrt{10}}$$

(ii) $Q = -11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

$$A = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix} \xrightarrow{(A - \lambda I)} \begin{bmatrix} -11-\lambda & 42 \\ 42 & 24-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (\lambda + 11)(\lambda - 24) - 42^2 = \lambda^2 - 13\lambda - 2028$$

$$\underbrace{\lambda_1 = 52}_{\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}} \qquad \underbrace{\lambda_2 = -39}_{\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}}$$

$$Q = 52 . y_1^2 - 39y_2^2 = 156 \rightarrow \frac{y_1^2}{3} - \frac{y_2^2}{4} = 1$$ ← hyperbola.

Normalized eigenvectors

$$\begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}, \begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{2y_1 - 3y_2}{\sqrt{13}} \qquad x_2 = \frac{3y_1 + 2y_2}{\sqrt{13}}$$

Q7: $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ $\quad x_1=\begin{bmatrix}1\\2\\-1\end{bmatrix}$ $\quad x_2=\begin{bmatrix}-2\\1\\0\end{bmatrix}$ $\quad x_3=\begin{bmatrix}3\\0\\1\end{bmatrix}$ $\qquad \lambda_1=5$, $\lambda_2=\lambda_3=-3$

(i) $x_1.x_2=\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}-2\\1\\0\end{bmatrix}=-2+2+0=0 \rightarrow$ ORTHOGONAL

$$\left.\begin{aligned} x_1.x_3 &=\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}3\\0\\1\end{bmatrix}=3+0-1=2 \\ x_2.x_3 &=\begin{bmatrix}-2\\1\\0\end{bmatrix}\begin{bmatrix}3\\0\\1\end{bmatrix}=-6+0+0=-6\end{aligned}\right\} \text{NOT ORTHOGONAL}$$

(ii) $e_1=\dfrac{u_1}{\|u_1\|}$ $\quad \|u_1\| \rightarrow \sqrt{1+4+1}=\sqrt{6}$ $\qquad e_1=\begin{bmatrix}\sqrt{6}/6\\ \sqrt{6}/3\\ -\sqrt{6}/6\end{bmatrix}$

$e_2=\dfrac{u_2}{\|u_2\|}$ $\rightarrow \|u_2\|=\sqrt{4+1}=\sqrt{5}$ $\qquad e_2=\begin{bmatrix}-\frac{2\sqrt{5}}{5}\\ \sqrt{5}/5\\ 0\end{bmatrix}$

$e_3=\dfrac{u_3}{\|u_3\|}$ $\qquad u_3=\begin{bmatrix}3\\0\\1\end{bmatrix}-\begin{bmatrix}1/3\\2/3\\-1/3\end{bmatrix}-\begin{bmatrix}12/5\\-6/5\\0\end{bmatrix}=\begin{bmatrix}4/15\\8/15\\4/3\end{bmatrix}$

$\|u_3\|=\dfrac{4\sqrt{30}}{15}$ $\qquad e_3=\begin{bmatrix}\sqrt{30}/30\\ \sqrt{30}/15\\ \sqrt{30}/6\end{bmatrix}$

Bonus: $A=[a_{jk}]=\begin{bmatrix}0.1 & 0.5 & 0\\ 0.8 & 0 & 0.4\\ 0.1 & 0.5 & 0.6\end{bmatrix}$

$p=A.p$

$$\left.\begin{aligned} p_1 &= 0.1p_1+0.5p_2 \\ p_2 &= 0.8p_1+0.1p_2+0.5.p_3 \\ p_3 &= 0.1p_1+0.5p_2+0.6p_3\end{aligned}\right\} \begin{aligned} &(A-I)p=0 \\ &|A-I|=0\end{aligned}$$

$0.4\,p_1.p_2.p_3-0.1p_2^2-0.6p_1p_3-0.5p_2p_3=0$

$(0.2p_1-0.5p_2)(2p_3-0.6p_1)=0$

$p = A.p \quad , \lambda = 1$

$A.p = (0.1 + 0.5 + 0, \; 0.8 + 0.1 + 0.5, \; 0.6 + 0.6 + 1)$

$A.p = (0.6, 1.4, 2.2)$

Normalized

$p = (0.273, 0.636, 1)$

Limit of this sequence is : $p = (0.5, 0.75, 1)$