Take Home Exam 6
Matrix Algebra

Q1. (*Eigenvalues and Eigenvectors*) Find the eigenvalues and the corresponding eigenvectors for the following matrices.

(i) $\begin{bmatrix} 4 & -6 \\ 3 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 12 \\ 0 & 0 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

Q2. (*Trace*). The sum of the main diagonal entries, called the trace of A, equals the sum of the eigenvalues of $\boldsymbol{A}$.

Using the following 2x2 matrix, show that the sum of the eigenvalues of $\boldsymbol{A}$ is equal to the sum of the diagonal elements a_{11} and a_{22}.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Q3. (*Symmetric/Skew-Symmetric/Orthogonal*) Are the following matrices symmetric, skew-symmetric, or orthogonal (orthonormal)?

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}$ (iv) $\begin{bmatrix} 1/9 & 4/9 & 8/9 \\ -4/9 & -7/9 & 4/9 \\ 8/9 & -4/9 & 1/9 \end{bmatrix}$

Q4. (*Similar Matrices*) For the given $\boldsymbol{A}$ and $\boldsymbol{P}$,

$$\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad \boldsymbol{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

a) Determine $\widehat{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\,\boldsymbol{P}$, the similar matrix of $\boldsymbol{A}$.

b) Show that $\boldsymbol{A}$ *and* $\widehat{\boldsymbol{A}}$ have the same eigenvalues.

c) Show that if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$ is an eigenvector of $\widehat{\boldsymbol{A}}$.

Q5. (*Diagonal Matrices*) Find an Eigenbases (a basis of eigenvectors) and diagonalize.

(i) $\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (ii) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Q6. (*Quadratic form and Coordinate Transformation*) What kind of geometric place is defined by the following quadratic form? Transform it to principal axes. Express in simplified terms of a new coordinate system.

(i) $\boldsymbol{Q} = x_1^2 + 6x_1x_2 + 9x_2^2 = 10$

(ii) $\boldsymbol{Q} = -11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

Q7. (*Gram-Schmitt Orthogonalization*) Consider the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The eigenvalues have been determined as $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$. The corresponding eigenvectors are

$$\boldsymbol{x_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{x_2} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x_1} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

(i) Are these orthogonal?

(ii) If not, find an orthogonal set based on these vectors using Gram-Schmidt Orthogonalization algorithm.

Q8. (Bonus) (Leontief Matrix) Suppose that *three* products are interrelated so that their outputs are used as inputs by themselves, according to the 3x3 *consumption matrix*, so called Leontief input-output model. (It is very commonly used matrix in economics)

$$\boldsymbol{A} = [a_{jk}] = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$$

Where a_{ij} is the fraction of the product of k used in product j. Let p_j be the price charged for product j.

A problem is to find prices so that for each product, total expenditures equal to the total income. This leads to $\boldsymbol{p} = \boldsymbol{A}\,\boldsymbol{p}$, where

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Find a solution for $\boldsymbol{p}$ with nonnegative p_1, p_2, p_3.

(Hint: Show that $\lambda = 1$ *is* an eigenvalue. Then consider $\boldsymbol{A}\,\boldsymbol{p} = \lambda\,\boldsymbol{p}$ with $\lambda = 1$. Hence $\boldsymbol{p}$ is an eigenvector corresponding to $\lambda = 1$.)