HOMEWORK 8 ANSWER KEY - EEE281

Q1:

a) $y^3 \cdot y' - x^2 = 0$

$y^3 \cdot y' = x^2$

$y^3 \cdot \frac{dy}{dx} = x^2$

$\int y^3 \cdot dy = \int x^2 \cdot dx$

$\frac{y^4}{4} = \frac{x^3}{3} + C$

$\boxed{y = \left[4\left(\frac{x^3}{3} + c\right)\right]^{1/4}}$

b) $y' = (\sec y)^2$

$\frac{dy}{dx} = (\sec y)^2$

$\frac{dy}{(\sec y)^2} = dx = \frac{dy}{\left(\frac{1}{\cos y}\right)^2} = (\cos^2 y)\,dy$

$\int dx = \int \cos^2 y \, dy$

$\cos 2a = 2\cos^2 a - 1$

$\frac{\cos 2a + 1}{2} = \cos^2 a$

$\int \frac{1 + \cos 2y}{2} \cdot dy = \int dx = \int \frac{1}{2} dy + \frac{1}{2}\int \cos 2y \, dy$

$x + C = \frac{y}{2} + \frac{\sin 2y}{4}$ ⇒ General Solution

c) $y' = 4 \cdot e^{4x-1} \cdot y^2$

$\frac{dy}{dx} = 4e^{4x-1} \cdot y^2$

$\frac{1}{y^2} \cdot dy = 4e^{4x-1} \cdot dx$

$\int \frac{1}{y^2} \cdot dy = 4\int e^{4x-1} dx$

$-\frac{1}{y} = \frac{4e^{4x-1}}{4} + C$

$-\frac{1}{y} = e^{4x-1} + C$

$\frac{1}{y} + e^{4x-1} = -C = A$

Q2: *) By setting $u = y/x$

a) $x.y' = y^2 + y$

$y = u.x \Rightarrow y' = u'.x + u$

$x(u'.x + u) = (ux)^2 + ux$

$u'.x^2 + \cancel{u.x} = u^2.x^2 + \cancel{ux}$

$u' = u^2$ if $(x \neq 0)$

$\frac{du}{dx} = u^2 \Rightarrow \int \frac{1}{u^2} \cdot du = \int dx \Rightarrow -\frac{1}{u} = x + C$

$\frac{1}{u} = -x - C \Rightarrow u = -\frac{1}{x+C} = \frac{y}{x} \Rightarrow y = -\frac{x}{x+C}$

For: $x.y' = y^2 + y$, $y = \frac{-x}{x+C}$ //

b) $x.y' = x + y$

$u = \frac{y}{x} \Rightarrow y = u.x$

$y' = u'.x + u$

$x(u'.x + u) = x + ux$

$x^2.u' + \cancel{x.u} = x + \cancel{ux}$

$x^2.u' = x$ if $(x \neq 0)$

$u' = 1/x$

$\frac{du}{dx} = \frac{1}{x} \rightarrow u = \ln x + c$

$\frac{y}{x} = \ln x + c \Rightarrow y = x.\ln x + x.c$

c) $x.y' = y + 3.x^4.\cos^2\left(\frac{y}{x}\right)$; $y(1) = 0$

$x(u'.x + u) = u.x + 3x^4.\cos^2(u)$

$x^2.u' + \cancel{u.x} = \cancel{u.x} + 3x^4.\cos^2(u)$

$x^2.u' = 3x^4.\cos^2(u)$

$u' = 3x^2.\cos^2(u)$

$\frac{du}{dx} = 3x^2.\cos^2(u)$

$\int \frac{1}{\cos^2(u)} du = \int 3x^2.dx$

$\tan(u) = 3\left(\frac{x^3}{3}\right) + C$

$\tan(u) = x^3 + C$

$\tan(0) = 1 + C = 0$

$C = -1$

$\tan\left(\frac{y}{x}\right) = x^3 - 1$

Q3: a) $y' = (y+4x)^2$; Set $u = y+4x$

$u = y+4x$

$\frac{du}{dx} = \frac{dy}{dx} + \frac{(4x)}{dx}$ $\Big\}$ $\frac{du}{dx} = \frac{dy}{dx} + 4$ $\Big\}$ $y' = u' - 4$

$u' - 4 = u^2 = y'$ $\Big\{$ $\frac{du}{dx} = u^2 + 4$ $\Big\}$ $\int \frac{du}{u^2+4} = \int dx + c$

$\frac{du}{dx} - 4 = u^2$ $\quad \frac{du}{u^2+4} = dx$ $\quad \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) = x + C$

$\frac{1}{2}\tan^{-1}\left(\frac{y+4x}{2}\right) = x + C$ //

b) $y' = (x+y-2)^2$, $y(0) = 2$, Set $u = x+y-2$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$ $\Big\}$ $\frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{du}{dx} - 1 = u^2$ $\Big\}$ $\frac{du}{dx} = u^2 + 1$ $\Big\}$ $\tan^{-1}(u) = x + C$

$\tan^{-1}(x+y-2) = x + C$

Q4: a) $2xy.dx + x^2 dy = 0$

$M(x,y).dx + N(x,y)dy = 0$

$\frac{\partial M}{\partial y} = M(x,y) \qquad \frac{\partial N}{\partial x} = N(x,y)$

Sol(a):

$M(x,y) = 2xy \qquad N(x,y) = x^2$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$

$2x = 2x \Rightarrow$ Exact

$\int 2xy.dx = x^2 y + g(y)$

$\int x^2 dy = x^2 y + h(x)$

$x^2 y + g(y) = x^2 y + h(x)$

$x^2 y = C$

b) $x^3 . dx + y^3 . dy = 0$

$M(x,y) = x^3 \qquad N(x,y) = y^3$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0 \Rightarrow$ Exact Solution

$$\left\{ \begin{array}{l} \int x^3 . dx = \frac{x^4}{4} + g(y) \\ \int y^3 dy = \frac{y^4}{4} + h(x) \end{array} \right.$$

$$\left(\frac{1}{4}\right)x^4 + \left(\frac{1}{4}\right)y^4 = C$$

c) $\underbrace{(\sin x . \cos y)}_{M(x,y)} dx + \underbrace{(\cos x . \sin y)}_{N(x,y)} dy = 0$

$\frac{\partial M}{\partial y} = 0 + \sin x . (-\sin y) \qquad \frac{\partial N}{\partial x} = (-\sin x) \sin y + 0$

$\frac{\partial M}{\partial y} = -\sin x . \sin y \longrightarrow$ Exact ✓

$\int \sin x . \cos y . dx = -\cos(x) . \cos(y) + g(y)$

$\int \cos x . \sin y . dy = -\cos(x) . \cos(y) + h(x)$

$$-\cos(x) . \cos(y) = C$$

d) $e^{3\theta} (dr + 3r . d\theta) = 0$

$M(\theta, r) = e^{3\theta} \qquad N(\theta, r) = e^{3\theta} . 3r$

$\frac{\partial M}{\partial \theta} = 3e^{3\theta}$

Exact

$\frac{\partial N}{\partial r} = 3e^{3\theta}$

$\int e^{3\theta} . dr = r . e^{3\theta} + g(\theta)$

$\int 3r . e^{3\theta} . d\theta = 3r \int e^{3\theta} d\theta = r . e^{3\theta} + h(r)$

$$C = r . e^{3\theta}$$

Q5. a) $\underbrace{3(y+1)\,dx = 2x \cdot dy}$; $F = (y+1)\,x^{-4}$

$M = 3(y+1)$ $\quad$ $N = -2x$

$\frac{\partial M}{\partial y} = 3$ $\quad$ $\frac{\partial N}{\partial x} = -2$ $\quad \Rightarrow \quad 3 \neq -2$ so, it is not an exact

$\left[3(y+1)dx - 2x \cdot dy\right](y+1)\,x^{-4} = 0$

$3(y+1)^2 \cdot x^{-4}dx - 2x^{-3}(y+1)dy = 0$

$M = 3(y+1)^2 \cdot x^{-4}$ $\quad$ $N = -2x^{-3}(y+1)$

$\frac{\partial M}{\partial y} = 6(y+1)\,x^{-4}$ $\quad$ $\frac{\partial N}{\partial x} = 6x^{-4}\cdot(y+1)$ $\Rightarrow$ it is an exact

$\int 3(y+1)^2 x^{-4}\,dx = 3(y+1)^2 \int x^{-4}dx = \frac{3}{-3}(y+1)^2 x^{-3} = -(y+1)^2 x^{-3}$

$\int -2x^{-3}(y+1)dy = -2x^{-3}\int (y+1)dy$

$\hookrightarrow = \frac{-2x^{-3}}{2}\cdot(y+1)^2 = -x^{-3}\cdot(y+1)^2 + c$

$-x^{-3}\cdot(y+1)^2 = -x^{-3}(y+1) + c$

$-(y+1)^2 = c \cdot x^3$

b) $y.dx + |y + \tan(x+y)| dy = 0 \qquad F = \cos(x+y)$

$M = y \qquad N = y + \tan(x+y)$

$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = \sec^2(x+y) \Rightarrow$ Not exact

Multiplying $\cos(x+y)$ on both sides.

$M = y.\cos(x+y) \qquad N = y.\cos(x+y) + \sin(x+y)$

$\frac{\partial M}{\partial y} = -y.\sin(x+y) + \cos(x+y) \qquad \frac{\partial N}{\partial x} = -y.\sin(x+y) + \cos(x+y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Exact

$\int y.\cos(x+y)\,dx = y\int \cos(x+y)\,dx = y.\sin(x+y)$

$\int (y.\cos(x+y) + \sin(x+y))\,dy \Rightarrow y.\sin(x+y) = c$

Q6: a) $(2xy.dx + dy)e^{x^2} = 0 \quad ; \; y(0) = 2$

$M = 2xy.e^{x^2} \qquad N = e^{x^2}$

$\frac{\partial M}{\partial y} = 2x.e^{x^2} \qquad \frac{\partial N}{\partial x} = 2x.e^{x^2}$

$\frac{\partial u}{\partial x} = 2xy.e^{x^2} \qquad \frac{\partial u}{\partial y} = y.e^{x^2}$

$\int du = \int 2xye^{x^2}dx$

$u = y.e^{x^2} + h(y)$

$u' = e^{x^2} + h'(y)$

$h'(y) = 0$

$y(0) = 2 \rightarrow$ Initial condition

$2e^0 + h(y) = 2$

$y.e^{x^2} = 2 \rightarrow$ Particular Solution

b) $(a+1).y.dx + (b+1)x.dy = 0$; $y(0)=1$; $F = x^a . y^b$

$M = (a+1).y$ $\qquad$ $N = (b+1).x$

$\frac{\partial M}{\partial y} = a+1$ $\qquad$ $\frac{\partial N}{\partial x} = b+1$ $\quad$ $a+1 \neq b+1 \rightarrow$ Not Exact

multiplying by $F = x^a . y^b$

$M = (a+1).y^{b+1}.x^a$ $\qquad$ $N = (b+1).y^b.x^{a+1}$

$\frac{\partial M}{\partial y} = (b+1)(a+1).y^b.x^a$ $\qquad$ $\frac{\partial N}{\partial x} = (b+1)(a+1).x^a.y^b$ $\quad$ Exact

$\frac{\partial u}{\partial x} = (x^{a+1})(a+1).y^b$ $\qquad$ $\frac{\partial u}{\partial y} = (x^{a+1})y^b.(b+1) + h'(y)$

$u = \int (a+1)(x^{a+1})y^b\,dx$ $\qquad$ $\frac{\partial u}{\partial y} = (b+1)(x^{a+1}).y^b$ $\quad$ $h'(y) = 0$

$u = \frac{a+1}{a+2}.(x^{a+2})y^b + \underbrace{h(y)}_{\text{must be constant}}$

$u = \frac{a+1}{a+2}(x^{a+2})y^b + c \longrightarrow$ initial condition $\rightarrow$ $c=1$

Particular solution $\Rightarrow$ $\frac{a+1}{a+2}(x^{a+2})y^b + 1$

Q7: $(ax+by)dx + (kx+ly)dy = 0$

$M = ax+by$ $\qquad$ $N = kx+ly$

$\frac{\partial M}{\partial y} = b$ $\qquad$ $\frac{\partial N}{\partial x} = k$ $\quad$ Exact: $\boxed{b=k}$

$\int (ax+by)dx = \frac{a}{2}x^2 + by.x + g(y) = F(x,y)$

$\frac{\partial F}{\partial y} = N(x,y)$ $\qquad$ $\frac{\partial\left(\frac{ax^2}{2} + byx + g(y)\right)}{\partial y} = k.x + l.y$

$g(y) = \frac{1}{2}y^2 + C$

$F(x,y) = \frac{a}{2}.x^2 + bxy + \frac{1}{2}y^2 + C$