13.1 Complex Numbers COMPLEX NUMBERS

Definition:

Consider the following quadratic equation:

$$z^2 - 2z + 2 = 0$$

Determine the roots:

$$az^2 + bz + c = 0$$

$$z_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = -2, c = 2$$

$$z_{1,2} = \frac{2 \mp \sqrt{4 - 4(1)(2)}}{2} = \frac{2 \mp \sqrt{-4}}{2}$$

$$= \frac{2 \mp 2\sqrt{-1}}{2} = 1 \mp \sqrt{-1}$$

So the roots are not real, they are complex numbers

Complex numbers are introduced by Girolamo Cardano (1501–1576)

The term Complex Number is first mentioned by Carl Friedrich Gauss (   ).

Define $i = \sqrt{-1}$, then

$$z_{1,2} = 1 \mp i$$

In general

$$z = x + iy$$

x : Real part

y = Imaginary part

$$z = x + iy = (x, y)$$

Addition :

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication :

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= x_1 x_2 + i x_2 y_1 + i x_1 y_2 + \underbrace{i^2}_{(-1)} y_1 y_2$$

$$= x_1 x_2 - y_1 y_2 + i(x_2 y_1 + x_1 y_2).$$

Subtraction :

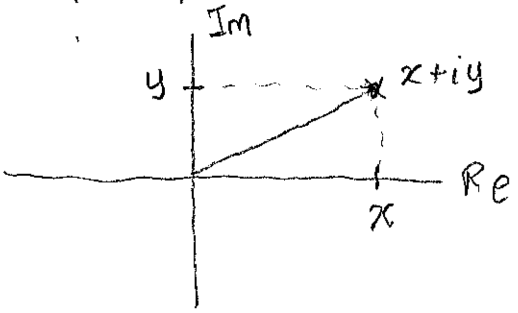
$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = x_1 - x_2 + i(y_1 - y_2)$$

Division

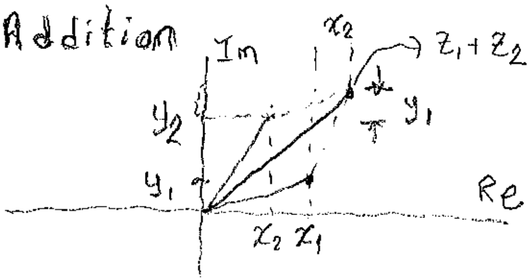
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \, (x_2 - iy_2) = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 - i^2 y_2^2} = \frac{x_1 x_2 + y_1 y_2 + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

Complex plane

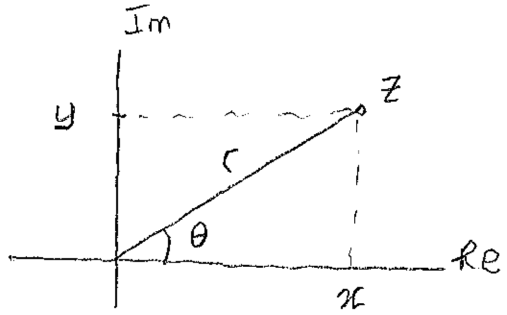


Addition



$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

13.2 Polar form



$$r = \sqrt{x^2+y^2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$z = x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Euler's Formula:

$$e^{j\theta} = \cos\theta + i\sin\theta$$

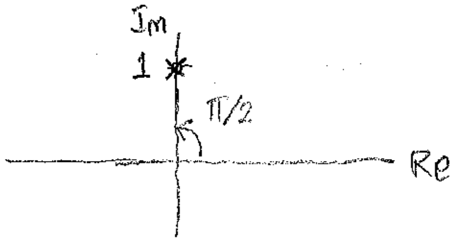
Then

$$z = re^{j\theta}$$

Polar form is very useful in multiplication, division, exponential operations.

<u>Example 1</u>

$$z = \sqrt{i}$$



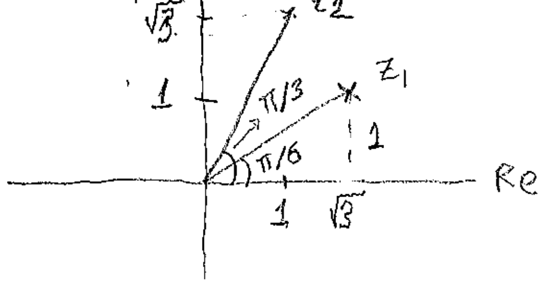
$$i = 0+i = e^{j\frac{\pi}{2}}$$

$$z = \sqrt{e^{j\frac{\pi}{2}}} = \left(e^{j\frac{\pi}{2}}\right)^{1/2}$$

$$= e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$$

<u>Example 2</u>



$$z_1 = \sqrt{3}+i$$

$$z_2 = 1+\sqrt{3}\,i$$

Determine $z = z_1 \cdot z_2$

$$z_1 = re^{j\theta}$$

$$z_1 : r = \sqrt{x^2+y^2} = \sqrt{3+1} = 2, \quad \theta = 30^\circ = \frac{\pi}{6} \Rightarrow z_1 = 2e^{j\frac{\pi}{6}}$$

$$z_2 : r = \sqrt{1+3} = 2, \quad \theta = 60^\circ = \frac{\pi}{3}$$

$$z_1 \cdot z_2 = (2e^{i\frac{\pi}{6}})(2e^{i\frac{\pi}{3}}) = 4e^{i(\frac{\pi}{6}+\frac{\pi}{3})} = 4e^{i\frac{3\pi}{6}} = 4e^{i\frac{\pi}{2}}$$

$$= 4i$$

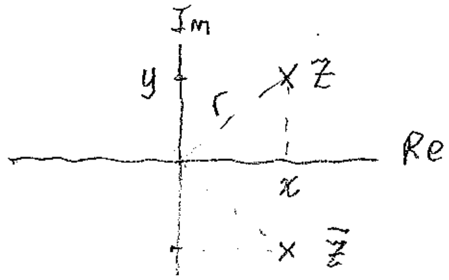
Check:

$$z_1 z_2 = (\sqrt{3}+i)(1+\sqrt{3}i) = \sqrt{3} + 3i + i + \sqrt{3}\underbrace{i^2}_{=-1}$$

$$= \sqrt{3} + 4i - \sqrt{3} = 4i$$

Complex conjugate

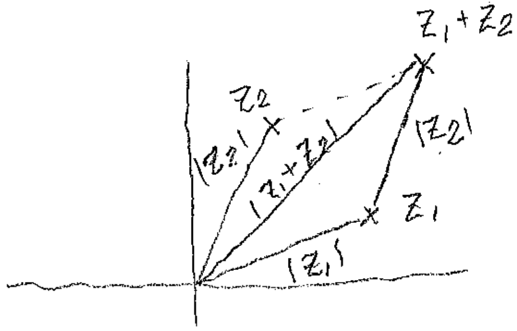
$$z = x + iy \Rightarrow \bar{z} = x - iy$$



$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 - \cancel{ixy} + \cancel{ixy} - \underbrace{i^2}_{-1} y^2$$
$$= x^2 + y^2$$
$$= r^2$$

$$r = |z| = \sqrt{x^2+y^2}$$

Triangular inequality



$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Example: $z_1 = \sqrt{3} + i \Rightarrow |z_1| = 2$

$z_2 = 1 + \sqrt{3}i \Rightarrow |z_2| = 2$

$z_1 + z_2 = \sqrt{3} + 1 + i(1+\sqrt{3})$

$|z_1 + z_2| = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}+1)^2} = \sqrt{2(\sqrt{3}+1)^2} = (\sqrt{3}+1)(\sqrt{2})$

$\cong 3.86$

$|z_1 + z_2| \leq |z_1| + |z_2|$

$3.86 \leq 2 + 2$ ✓

Integer Powers of Z (De Moivre's Formula)

Powers of Z:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin\theta)$$

$r = 1$:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta$$

Roots

$$w = \sqrt[n]{z} \Rightarrow$$

$$z = re^{j\theta} = re^{j(\theta \mp 2\pi k)}$$

$$w = \sqrt[n]{z} = z^{1/n} = r^{1/n} e^{j\frac{1}{n}(\theta \mp 2\pi k)}$$

$$= r^{1/n}\left[\cos\left(\frac{\theta \mp 2\pi k}{n}\right) + i\sin\left(\frac{\theta \mp 2\pi k}{n}\right)\right] \quad k = 0, 1, \ldots$$

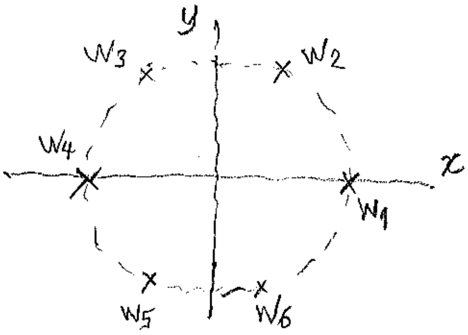
Example

$$(1+i\sqrt{3})^3 = (2e^{j\frac{\pi}{3}})^3 = 2^3 e^{j\pi} = 8(\underbrace{\cos\pi}_{-1} + \underbrace{i\sin\pi}_{0}) = -8$$

Example

$$w^5 = 1 \Rightarrow w = \sqrt[6]{1}$$

$$w = (1e^{i2\pi n})^{1/6} = 1^{1/6} \cdot e^{i\frac{2\pi n}{6}} = 1 \cdot e^{i\frac{\pi}{3}n}$$



$n = 0 \rightarrow w_1 = 1$

$n = 1 \Rightarrow w_2 = e^{j\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$n = 2 \Rightarrow w_3 = e^{i\frac{4\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

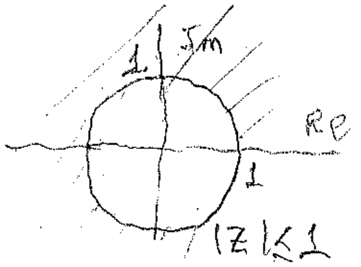
$n = 3 \Rightarrow w_4 = e^{i\pi} = \cos\pi + i\sin\pi = -1$

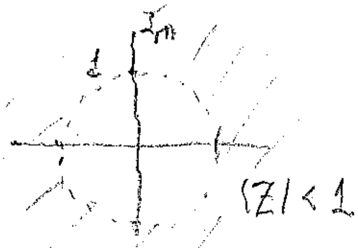
$n = 4 \rightarrow w_5 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

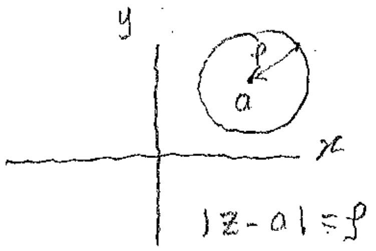
$n = 5 \Rightarrow w_6 = e^{i\frac{5\pi}{3}} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

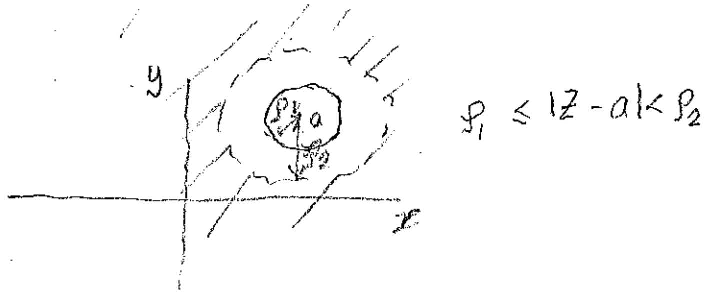
13.3. Complex Functions (619)

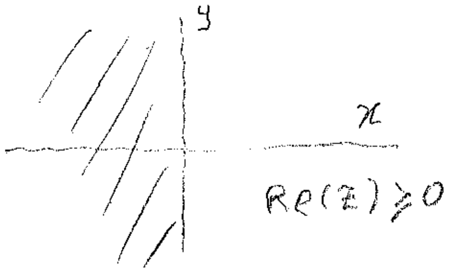
Complex domain : The values of z over which the function f is to defined











$\underline{z \in D.}$

Complex function : A function of z defined over a domain D.

$$w = f(z) \quad z \in D, \quad (w \text{ is complex value})$$

Range : The values that w can take.

$$W = f(z) = u(x,y) + i\,v(x,y)$$

<u>Example:</u> $w = f(z) = z^2 + 1$.

Evaluate w at $z = 1 + i$

$$w = (1+i)^2 + 1 = 1 + 2i + i^2 + 1 = \cancel{1} + 2i - \cancel{1} + 1$$
$$= 1 + 2i$$

$u \qquad v$

Limit

The function $f(z)$ has the limit ℓ as z appraches z_0,

$$\lim_{z \to z_0} f(z) = \ell$$

if $z \neq z_0$

$$|z - z_0| < \delta \Rightarrow |f(z) - \ell| < \epsilon,$$ z may approch to z_0 from any direction

Continuity

The function $f(z)$ is said to be continuous if at $z = z_0$, $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Example :

$$f(z) = z^2 - 1$$

$$\lim_{z \to 1} f(z) = 0 \quad (\text{Limit at } z = 1)$$

$$f(1) = 0 \quad (f(z) \text{ is continuous at } z = 1$$

Example

$$g(z) = \frac{z^2 - 1}{z - 1}$$

$$\lim_{z \to 1} f(z) = \frac{(z-1)(z+1)}{z-1} = 2$$

However
$g(1)$ is not defined. $\Rightarrow$ $g(z)$ is not continous at $z = 1$

Derivative

The derivative of f at a point z_0 $(\Delta z = z - z_0)$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example: $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\cancel{z} + \cancel{\Delta} z - \cancel{z})(z + \Delta z + z)}{\cancel{\Delta z}} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Differentiation Rules:

$(cf)' = cf'$ (c is constant)

$(f+g)' = f' + g'$

$(f \cdot g)' = f'g + fg'$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example: Not differentiable

$f(z) = \bar{z}$

$\Delta z = \Delta x + i \Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{x + \Delta x + i(y + \Delta y)} - \overline{x + iy}}{\Delta x + i \Delta y} = \frac{\cancel{x} + \Delta x - i(\cancel{y} + \Delta y) - (\cancel{x} - i\cancel{y})}{\Delta x + i \Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y} \begin{cases} 1 & \Delta y = 0 \\ -1 & \Delta x = 0 \end{cases}$$

! Not differentiable

Analytic Functions (Holomorphic functions)

For $z \in D$, $f(z)$ is defined and differentiable, $f(z)$ is said to be analytic over the domain D

Polynomials

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

Rationals

$$f(z) = \frac{g(z)}{h(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m}$$

13.4 Cauchy-Rieman Equations (625)

If f is analytic in a domain D if and only if the first partial derivatives satisfy the following equations so called Cauchy-Rieman Equations

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example:

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$u: x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$

$v = 2xy \Rightarrow \frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$

Example: $f(z) = \bar{z} = x - iy$

$u = x \qquad u_x = 1 \quad ? \quad u_y = 0$

$v = -y \qquad v_x = 0 \qquad v_y = -1$

Cauchy-Rieman equations are not satisfied. $f(z) = \bar{z}$ is not analytic as shown before.

Example: $f(z) = e^x(\cos y + i \sin y)$

$u = e^x \cos y \Rightarrow u_x = e^x \cos y \quad \checkmark \quad u_y = -e^x \sin y$

$v = e^x \sin y \Rightarrow v_x = e^x \sin y \qquad v_y = e^x \cos y.$

$f(z)$ is analytic for all z.

Example: $|f(z)| = k \Rightarrow f(z)$ is constant

$$|u + iv| = k \Rightarrow u^2 + v^2 = k^2 \qquad \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$$

Partial derivative w.r.t x:

$$\cancel{2}uu_x + \cancel{2}vv_x = 0 \rightarrow u^2u_x + uv\underbrace{v_x}_{=-u_y} = 0$$

Partial derivative w.r.t y:

$$\cancel{2}uu_y + \cancel{2}vv_y = 0$$

Assume $u_x = v_y$ in (2)

$$uu_y + vu_x = 0 \qquad \rightarrow uvu_y + v^2u_x = 0$$

$$(u^2 + v^2)u_x = 0$$

Similarly:

$$(u) \qquad uv_y + vv_x = 0 \Rightarrow u^2v_y + \cancel{uvv_y} = 0$$

$$(v) \qquad -uv_x + vv_y^2 = 0 \qquad -\cancel{uvv_x} + v^2v_y = 0$$

$$(u^2 + v^2)v_y = 0$$

$u_x = v_y = 0 \Rightarrow u, v$ constant

$\Rightarrow u_x \neq 0, v_y \neq 0 \Rightarrow u = v = 0$

Cauchy Rieman Equations in Polar form:

$$f(z) = u(r,\theta) + i v(r,\theta)$$

$$z = r(\cos\theta + i\sin\theta) \Rightarrow x = r\cos\theta,\ y = r\sin\theta$$

$$U_x = V_y \quad ; \quad U_y = -V_x$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\,\frac{1}{\left(\frac{\partial x}{\partial r}\right)} + \frac{\partial u}{\partial \theta}\cdot\frac{1}{\left(\frac{\partial x}{\partial \theta}\right)}$$

$$u_x = u_r\frac{1}{\cos\theta} - u_\theta\frac{1}{r\sin\theta}$$

$$u_y = u_r\frac{1}{\sin\theta} + u_\theta\frac{1}{r\cos\theta}$$

$$v_x = v_r\frac{1}{\cos\theta} - v_\theta\frac{1}{r\sin\theta} \quad ; \quad v_y = v_r\frac{1}{\sin\theta} + v_\theta\frac{1}{r\cos\theta}$$

$$u_x = v_y \Rightarrow u_r\frac{1}{\cos\theta} - u_\theta\frac{1}{r\sin\theta} = v_r\frac{1}{\sin\theta} + v_\theta\frac{1}{r\cos\theta} \qquad (1)$$

$$u_y = -v_x \Rightarrow u_r\frac{1}{\sin\theta} + u_\theta\frac{1}{r\cos\theta} = -v_R\frac{1}{\cos\theta} + v_\theta\frac{1}{r\sin\theta} \qquad (2)$$

$$\left(u_r - \frac{v_\theta}{r}\right)\frac{1}{\cos\theta} - \left(\frac{u_\theta}{r} + v_r\right)\frac{1}{\sin\theta} = 0 \qquad (1a)$$

$$\left(u_r - \frac{v_\theta}{r}\right)\frac{1}{\sin\theta} + \left(\frac{u_\theta}{r} + v_R\right)\frac{1}{\cos\theta} = 0 \qquad (2a)$$

In order (1a) and (2a) to be satisfied for any θ:

$$u_r - \frac{v_\theta}{r} = 0 \Rightarrow u_r = \frac{v_\theta}{r}$$

$$\frac{u_\theta}{r} + v_r = 0 \Rightarrow v_r = -\frac{u_\theta}{r}$$

Example: $f(z) = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2(\cos 2\theta + i\sin 2\theta)$

$$\left.\begin{aligned} u = r^2\cos 2\theta &\Rightarrow u_r = 2r\cos 2\theta,\ u_\theta = r^2(-2\sin\theta) \\ v = r^2\sin 2\theta &\Rightarrow v_r = 2r\sin 2\theta,\ v_\theta = r^2(2\cos 2\theta) \end{aligned}\right\} \begin{aligned} u_r &= \tfrac{1}{r}v_\theta \ ✓ \\ v_r &= -\tfrac{1}{r}u_\theta \ ✓ \end{aligned}$$