Exact ODE

Total derivative

$u(x,y)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy$$

Example:

$$u = x + x^2 y^3 = C$$

$$du = (1+2xy^3)\, dx + 3x^2 y^2\, dy = 0$$

Then $y' = \frac{dy}{dx} = -\frac{1+2xy^3}{3x^2y^2}$

Assume

$$y' = -\frac{M(x,y)}{N(x,y)}$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \Rightarrow N(x,y)\,dy = -M(x,y)\,dx$$

or

$$M(x,y)\,dx + N(x,y)\,dy$$

is said to be exact ODE if

$$M(x,y) = \frac{\partial u}{\partial x}, \quad N(x,y) = \frac{\partial u}{\partial y}$$

Then we have

$$du = 0 \Rightarrow u(x,y) = 0$$

is a solution.

Check for exact ODE:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M = \frac{\partial u}{\partial x} \rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$N = \frac{\partial u}{\partial y} \rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 y}{\partial x \partial y}$$

Equal from continuity, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for an exact ODE.

Example: $\underbrace{\cos(x+y)}_{M}dx + \underbrace{[3y^2+2y+\cos(x+y)]}_{N}dy = 0$

Check for exact ODE:

$M = \cos(x+y) \Rightarrow \frac{\partial M}{\partial y} = -\sin(x+y)$

$N = 3y^2 + 2y + \cos(x+y) \Rightarrow \frac{\partial N}{\partial x} = -\sin(x+y)$

Equal so exact ODE

Solution:

$M = \frac{\partial u}{\partial x} = \cos(x+y) \Rightarrow u = \sin(x+y) + h(y)$

$\frac{\partial u}{\partial y} = N \Rightarrow \cancel{\cos(x+y)} + h'(y) = 3y^2 + 2y + \cancel{\cos(x+y)}$

$h'(y) = 3y^2 + 2y$

$h(y) = \int(3y^2+2y)dy$

$= y^3 + y^2 + c^*$

Then $u = \sin(x+y) + h(y) = c$

$= \sin(x+y) + y^2 + y^3 + c^* = c$

or

$\sin(x+y) + y^2 + y^3 = c$ (Implicit solution)

This typeofsolution is called <u>implicit solution</u>, since y is not expressed as a function of x.

If y is expressed in terms of x, it is explicit solution.

Example: Initial value is given.

$$(\cos y \sinh x + 1)\,dx - (\sin y \cosh x)\,dy = 0 \qquad \underline{y(1) = 2}$$

$$M = \cos y \sinh x \rightarrow \frac{\partial M}{\partial y} = -\sin y \sinh x$$

$$N = -\sin y \cosh x \rightarrow \frac{\partial N}{\partial x} = -\sin y \sinh x$$

Equal. So exact ODE

$$M = \frac{\partial u}{\partial x} = \cos y \sinh x + 1 \rightarrow u = \cos y \cosh x + x + h(y)$$

$$\frac{\partial u}{\partial y} = N \Rightarrow -\sin y \cosh x + h'(y) = -\sin y \cosh x$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = c^*$$

$$u(x,y) = \cos y \cosh x + x + c^* = c$$

$$\cos y \cosh x + x = c'$$

$$y(1) = 2 \Rightarrow x = 1,\ y = 2$$

$$(\cos 2)(\cosh 1) + 1 = c' \Rightarrow c' = 0.358$$

$$(-0.416)(1.543) + 1 = c'$$

Then

$$u(x,y) = \cos y \cosh x + x + 0.358 = 0$$

1.4 Exact ODEs. Integrating Factors

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} \cdot dy \quad \text{(Total differential of } u\text{)}$$

If $u(x,y) = c$ (constant) then $du = 0$.

Example: $u = x + x^2 y^3 = c$

$$du = (1 + 2x y^3) dx + 3x^2 y^2 dy = 0$$

$$3x^2 y^2 \, dy = -(1 + 2x y^3) dx$$

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2 y^2}.$$

$u(x,y) = c \rightarrow$ Implicit solution

Exact differential equation

$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow$ Exact DE

if

$$M(x,y)dx + N(x,y)dy$$

is exact; i.e this expression is differential

$$du = \underbrace{\frac{\partial u}{\partial x}}_{M(x,y)} \cdot dx + \underbrace{\frac{\partial u}{\partial y}}_{N(x,y)} \cdot dy$$

of some function $u(x,y)$. Then.

$du = 0 \Rightarrow u(x,y) = c$ (constant)

$$M = \frac{\partial u}{\partial x} \quad ; \quad N = \frac{\partial u}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad ; \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Using the continuity of two second partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example : $du = \underbrace{(1+2xy^3)}_{M}dx + \underbrace{3x^2y^2}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = 6xy^2 \qquad \frac{\partial N}{\partial x} = 6xy^2$$

Solution :

$$du = M(x,y)dx + N(x,y)dy$$

$$u = \int M dx + k(y)$$

$$\frac{\partial u}{\partial y} = N$$

Example : $du = \underbrace{(1+2xy^3)}_{M}dx + \underbrace{3x^2y^2}_{N}dy \rightarrow$ Exact !

$$u = \int (1+2xy^3)dx + k(y)$$

$$u = x + x^2y^3 + k(y)$$

$$\frac{\partial u}{\partial y} = N(x,y) = 3x^2y^2 + \frac{\partial k}{\partial y} = 3x^2y^2 \Rightarrow \frac{\partial k}{\partial y} = 0 \Rightarrow k = c'$$

Hence $\quad u(x,y) = x + x^2y^3 + c' = c''$

$$x^2 + x^2y^3 = c'' - c' = c \quad \text{(constant)}$$

Example :

Reduction to Exact Form (Integrating Factor)

In the case

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is not an exact differential equation, by multiply it with $F(x)$ or $G(y)$ or $H(x,y)$ it can be put in the form of exact ODE.

Example:

$$\underbrace{[e^{(x+y)} + ye^{y}]}_{M}dx + \underbrace{[xe^{y} - 1]}_{N}dy = 0 \qquad y(0) = -1$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= e^{(x+y)} + e^{y} + ye^{y} \\ \frac{\partial N}{\partial x} &= e^{y} \end{aligned}\right\} \text{Not exact.}$$

Now multiply both sides $e^{-y} \neq 0$.

$$[e^{(x+y)} + ye^{y}]e^{-y}\,dx + [xe^{-y} - 1]e^{-y}\,dy = 0$$

$$\underbrace{(e^{x} + y)}_{M}dx + \underbrace{(x - e^{-y})}_{N}dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned}\right\} \text{Exact!}$$

$$M = \frac{\partial u}{\partial x} = e^{x} + y \Rightarrow u = e^{x} + yx + h(y)$$

$$\frac{\partial u}{\partial y} = N \Rightarrow x + h'(y) = x - e^{-y} \Rightarrow h'(y) = -e^{-y}$$

$$h(y) = e^{-y}$$

Then $u = e^{x} + yx + e^{-y} = C$

$y(0) = -1 \Rightarrow e^{0} + (-1)(0) + e^{-y} = C \Rightarrow 1 + e = C = 3.72$

$e^{x} + yx + e^{-y} = 3.72$ is a solution.

First Order ODEs (Linear)

$$y' + p(x)y = r(x)$$

If a differential equation is brought into the above form. It is said First order Linear ODEs

Linear First Order ODE

$$y' + p(x)\,y = r(x)$$

Homogeneous
$r(x) = 0$

Nonhomogeneous
$r(x)$ exist

<u>Homogeneous Linear ODEs</u> (First order)

$$y' + p(x)y = 0$$

$$\frac{dy}{dx} = -p(x)y \Rightarrow \frac{dy}{y} = -p(x)dx$$

$$\ln y = \int -p(x)\,dx$$

$$y = e^{-\int p(x)dx} = ce^{-\int p(x)dx}$$

Example:

$$y' - 2xy = 0 \rightarrow y' = +2xy$$

$$\frac{dy}{y} = 2x\,dx \Rightarrow \ln y = x^2 + c^*$$

$$y = e^{x^2} \cdot e^{c^*} = ce^{x}$$

First order Linear Nonhomogeneous ODEs

$$y' + p(x)y = r(x)$$

Multiply both sides by $e^{\int p(x)dx}$:

$$\underbrace{y' e^{\int p(x)dx} + p(x) e^{\int p(x)dx} y}_{\frac{d}{dx}\left[e^{\int p(x)dx} y\right]} = r(x) e^{\int p(x)dx}$$

$$\frac{d}{dx}\left[e^{\int p(x)dx} y\right] = r(x) \cdot e^{\int p(x)dx}$$

$$d\left[e^{\int p(x)dx} y\right] = r(x)\,dx$$

Then

$$e^{\int p(x)dx} y = \int r(x)\,dx$$

$$\Rightarrow y = \frac{\int r(x)\,dx}{e^{\int p(x)dx}}$$

Example :

$$y' + \underbrace{(\tan x)}_{p} y = \underbrace{\sin 2x}_{r} \qquad y(0) = 1$$

$$e^{\int \tan x\,dx} = e^{(\ln \sec x)} = \sec x = \frac{1}{\cos x}$$

Multiply both sides by $\frac{1}{\cos x}$:

$$y' + \frac{\tan x}{\cos x} y = \frac{\sin 2x}{\cos x}$$

$$y' \frac{1}{\cos x} + \frac{\sin x}{\cos^2 x} y = \frac{2 \sin x \cos x}{\cos x}$$

$$\frac{d}{dx}\left(\frac{y}{\cos x}\right) = \frac{y' \cos x - (-\sin x) y}{\cos^2 x} = \frac{y'}{\cos x} + \frac{\sin x}{\cos^2 x} y$$

$$\frac{d}{dx}\left(\frac{y}{\cos x}\right) = 2 \sin x$$

$$\int d\left(\frac{y}{\cos x}\right) = \int 2 \sin x\,dx \Rightarrow \frac{y}{\cos x} = -2\cos x + c$$

$$y = -2\cos^2 x + c \cdot \cos x$$

$$y(0) = 1 \Rightarrow 1 = -2 + c \Rightarrow c = 3$$

$$y = -2\cos^2 x + 3\cos x$$

Linear ODE first order with constant Coefficients

$$y' + ax = r(x)$$

Superposition:

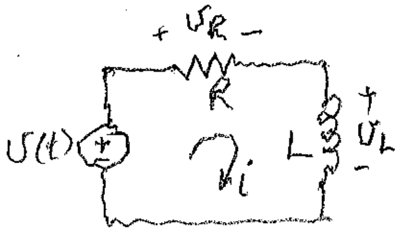
Homogeneous $y' + ax = 0 \rightarrow y_h = ce^{\lambda t}$.

Nonhom $y' + ax = r(x) \rightarrow y_p$ particular solution

$$y = y_h + y_p$$

$r(x)$	y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_0$
$k\cos\omega_0 t$	$K\cos\omega_0 t + M\sin\omega_0 t$

Example:



$$-v + v_R + v_L = 0$$

$$v_L + v_R = v$$

$$v_L = L\frac{di}{dt}, \quad v_R = iR$$

$$L\frac{di}{dt} + Ri = v(t) \qquad i(0) = 0$$

$$i' + \frac{R}{L} i = \frac{1}{L} v(t) \Rightarrow y' + \frac{R}{L} y = \frac{1}{L} v$$

i. Homogeneous solution:

$$i' + \frac{R}{L} i = 0 \qquad (y' + \frac{R}{L} y =$$

Assume $y_h = ce^{\lambda t} \Rightarrow y_h' = c\lambda e^{\lambda t}$

$$c\lambda e^{\lambda t} + \frac{R}{L} c e^{\lambda t} = 0 \Rightarrow c(\lambda + \frac{R}{L}) e^{\lambda t} = 0$$

$$\rightarrow \lambda = -\frac{R}{L} \Rightarrow y_h = ce^{-\frac{R}{L}t}$$

ii. Particular solution: Assume $v(t) = E$ (constant) $\Rightarrow y_p = C$

Then $y_p' = 0 \Rightarrow 0 + C = \frac{1}{L}E \rightarrow y_p = \frac{E}{L}$

Total Solution

$$y = y_p + y_h = Ce^{-\frac{R}{L}t} + \frac{E}{L}$$

$$y(0) = 0 \Rightarrow C + \frac{E}{L} = 0 \Rightarrow C = -\frac{E}{L}$$

$$i(t) = -\frac{E}{L}e^{-\frac{R}{L}t} + \frac{E}{L} = \frac{E}{L}(1 - e^{-\frac{R}{L}t})$$

Now assume $v(t) = \cos w_0 t$; $i' + \frac{R}{L}i = \frac{1}{L}v$

Then $\quad i_p = A\cos w_0 t + B\sin w_0 t$

$$i_p' = -Aw_0 \sin w_0 t + Bw_0 \cos w_0 t$$

$$i_p' + \frac{R}{L}i_p = \frac{1}{L}\cos w_0 t$$

$$-Aw_0 \sin w_0 t + Bw_0 \cos w_0 t + \frac{R}{L}A\cos w_0 t + B\frac{R}{L}\sin w_0 t = \frac{1}{L}\cos w_0 t$$

$$-Aw_0 + B\frac{R}{L} = 0 \quad \Rightarrow B = Aw_0\frac{L}{R}$$

$$Bw_0 + A\frac{R}{L} = \frac{1}{L} \qquad A\frac{L}{R}w_0^2 + A\frac{R}{L} = \frac{1}{L}$$

$$\cancel{\Rightarrow A = \frac{1}{1+w_0^2} \quad B = \frac{w_0}{1+w_0^2}} \qquad A\left(\frac{L^2w_0^2}{R} + R\right) = 1$$

$$\cancel{i_p(t) =} \qquad A = \frac{R}{L^2w_0^2 + R^2}$$

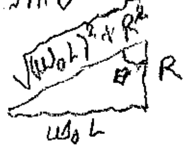
$$B = \frac{w_0\frac{L}{R}\cdot R}{L^2w_0^2 + R^2} = \frac{w_0 L}{w_0^2L^2 + R^2}$$

$$i_p = A\cos w_0 t + B\sin w_0 t$$

$$= \frac{R}{w_0^2L^2 + R^2}\cos w_0 t + \frac{w_0 L}{w_0^2L^2 + R^2}\sin w_0 t$$

$$= \frac{1}{\sqrt{(w_0L)^2 + R^2}}\left[\underbrace{\frac{R}{\sqrt{(w_0L)^2+R^2}}}_{\cos\theta}\cos w_0 t + \underbrace{\frac{w_0L}{\sqrt{(w_0L)^2+R^2}}}_{\sin\theta}\sin w_0 t\right]$$

$$= \frac{1}{\sqrt{(w_0L)^2 + R^2}}\cos(w_0 t - \theta)$$



$$\theta = \tan^{-1}\left(\frac{w_0L}{R}\right)$$

Then $\quad i = i_h + i_p = Ce^{-\frac{R}{L}t} + \frac{1}{\sqrt{(w_0L)^2 + R^2}}\cos(w_0 t - \theta)$

Example

$$i' + \frac{R}{L} i = \frac{1}{L} v(t)$$

Now assume $v(t) = t$

$$i_h = c e^{-\frac{R}{L} t}$$

Particular solution : $i_p = At + B$

$$i_p' = A$$

$$i' + \frac{R}{L} i = \frac{1}{L} t$$

$$A + \frac{R}{L}(At + B) = \frac{1}{L} t$$

$$\frac{R}{L} At + (A + \frac{R}{L} B) = \frac{1}{L} t \Rightarrow \frac{R}{L} A = \frac{1}{L} \Rightarrow A = \frac{1}{R}$$

$$\frac{1}{R} + \frac{R}{L} B = 0 \Rightarrow B = -\frac{1}{R} \frac{L}{R} = -\frac{L}{R^2}$$

$$i_p = At + B = \frac{1}{R} t - \frac{L}{R^2}$$

Then total solution :

$$i = c e^{-\frac{R}{L} t} + \frac{1}{R} t - \frac{L}{R^2}$$

Assume $i(0) = 0$

$$0 = c - \frac{L}{R^2} \Rightarrow c = \frac{L}{R^2}$$

$$i = \frac{L}{R^2} e^{-\frac{R}{L} t} + \frac{1}{R} t - \frac{L}{R^2} = \frac{L}{R}(e^{-\frac{R}{L} t} - 1) + \frac{1}{R} t$$