SYSTEM OF ODES

consider the second order linear ODE : $y = y(t)$; $\frac{dy}{dt} = y'$

(1) $y'' + 5y' + 4y = r(x)$ $\quad y(0) = 0, \; y'(0) = 1$

Homogeneous solution (when $r(x)$ is set to 0) should be determined first.

Homogeneous solution is the system behaviour when there is no input i.e., $r(x) = 0$.

Homogeneous solution :

$$y'' + 5y' + 4y = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad \text{(characteristic equation)}$$

$$\Delta = 25 - 16 = 9$$

$$\Rightarrow \lambda_{1,2} = \frac{-5 \mp 3}{2} \begin{cases} -4 \\ -1 \end{cases}$$

So homogeneous solution

$$y_h(x) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-4t} + c_2 e^{-}$$

(2) Now consider the first order linear ODE :

$$y' + 4y = r(t)$$

Homogeneous solution : $r(t) = 0$

$$y' + 4y = 0 \Rightarrow y' = -4y \Rightarrow \frac{dy}{dt} = -4y$$

$$\lambda + 4 = 0 \Rightarrow \lambda = -4 \qquad \frac{dy}{y} = -4dt$$

$$y_h = ce^{-4x}$$

$$\ln y = -4t + c'$$

$$y = ce^{-4t}$$

Now consider ODE (1) :

$$y'' + 5y' + 4y = 0 \qquad (1)$$

Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$

$y_1' = y' = y_2 \Rightarrow y_1' = y_2$

$y_2' = y'' = y_3 = -5y' - 4y = -5y_2 - 4y_1$

From (1)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$

$$\frac{d}{dt} \underline{y} = \underline{A}\, \underline{y}$$

Then the solution

$$\underline{y} = e^{\underline{A}t} \underline{c}$$

How can we determine if $\underline{A}$ is a matrix ?

Consider

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

Eigenvalues of $\underline{A}$:

Characteristic equation :

$$\det(\underline{A} - \lambda \underline{I}) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

$$-\lambda(-5-\lambda) - (-4) = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \rightarrow \lambda_1 = -4, \lambda_2 = -1$$

Same characteristic equation as obtained before for the ODE.

$$\lambda_1 = -4, \quad \lambda_2 = -1$$

Determine the corresponding eigen vectors

$\lambda_1 = -4$:

$$(\underline{A} - \lambda \underline{I})\, \underline{x}_1 = 0 \qquad \underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$(\underline{A} + 4\underline{I})\, \underline{x}_1 = 0 \Rightarrow \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$4x_1 + x_2 = 0$$

$$x_2 = -4, \quad x_1 = 1 \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Normalized : $\underline{x}_1 = \begin{bmatrix} 1/\sqrt{17} \\ -4/\sqrt{17} \end{bmatrix}$

$\lambda_2 = -1$

$$(\underline{A} - \lambda \underline{I})\, \underline{x}_2 = 0 \qquad \underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$(\underline{A} + \underline{I})\, \underline{x}_2 = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$x_1 + x_2 = 0$$

$$x_2 = -1, \quad x_1 = 1 \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalized : $\underline{x}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Now define

$$\underline{X} = [\underline{x}_1 \; \underline{x}_2] = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}, \quad \det(\underline{X}) = 3$$

$$\underline{X}^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix}$$

Remember :

$$\underline{D} = \underline{X}^{-1} \underline{A}\, \underline{X}$$

$$\underline{D} = \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 16 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4-16}{3} & 0 \\ 0 & \frac{-4+1}{3} \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that instead of $\underline{x}_1$ and $\underline{x}_2$; if we have used normalized eigenvectors (i.e., $\frac{\underline{x}_1}{\|\underline{x}_1\|}$ & $\frac{\underline{x}_2}{\|\underline{x}_2\|}$) the same result would be obtained. But then the property of

$$\underline{X}^{-1} = \underline{X}^T \quad \text{(orthogonality)}$$

would be satisfied. This property is used to simplify the quadratic form

$$Q = a x_1^2 + b x_1 x_2 + c x_2^2 .$$

Then

$$\underline{D} = \underline{X}^{-1} \underline{A} \underline{X} \Rightarrow \underline{X}\underline{D} = \underbrace{\underline{X}\underline{X}^{-1}}_{\underline{I}} \underline{A}\underline{X} = \underline{I}\underline{A}\underline{X} = \underline{A}\underline{X}$$

$$\underline{A}\underline{X} = \underline{X}\underline{D} \Rightarrow \underline{A}\underbrace{\underline{X}\underline{X}^{-1}}_{\underline{I}} = \underline{X}\underline{D}\underline{X}^{-1} \Rightarrow \underline{A} = \underline{X}\underline{D}\underline{X}^{-1}$$

$$\underline{A} = \underline{X}\underline{D}X^{-1}$$

$$\underline{A}^2 = \underline{A}\,\underline{A} = (\underline{X}\underline{D}\underbrace{\underline{X}^{-1})(\underline{X}}_{\underline{I}}\underline{D}\underline{X}^{-1}) = \underline{X}\underline{D}^2\underline{X}^{-1}$$

$$\underline{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad D^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\underline{D}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$e^{t} = \sum_{n=0}^{\infty} \frac{t}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

$$e^{\underline{A}t} = I + \underline{A}\,t + \frac{\underline{A}^2 t^2}{2!} + \frac{\underline{A}^3 t^3}{3!} + \dots$$

$$= \underline{X}\,\underline{X}^{-1} + \underline{X}\,\underline{D}\,X^{-1} t + \frac{1}{2!}\underline{X}\,\underline{D}^2\underline{X}^{-1} t^2 + \frac{1}{3!}\underline{X}\,\underline{D}^3\underline{X}^{-1} t^3 + \dots$$

$$= \underline{X}X^{-1} + \underline{X}(\underline{D}t)\underline{X}^{-1} + \underline{X}\frac{(\underline{D}t)^2}{2}\underline{X}^{-1} + X\left[\frac{(\underline{D}t)^3}{3!}\right]X^{-1} + \dots$$

$$= \underline{X}\left[I + \underline{D}t + \frac{(\underline{D}t)^2}{2!} + \frac{(\underline{D}t)^3}{3!} + \dots\right]X^{-1}$$

$$= \underline{X}\begin{bmatrix} 1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + & 0 \\ 0 & 1 + \lambda_2 t + \end{bmatrix} X^{-1}$$

$$= \underline{X}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} X^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3}e^{\lambda_1 t} & -\frac{1}{3}e^{\lambda_1 t} \\ \frac{4}{3}e^{\lambda_2 t} & \frac{1}{3}e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}e^{\lambda_1 t} + \frac{4}{3}e^{\lambda_2 t} & -\frac{1}{3}e^{\lambda_1 t} + \frac{1}{3}e^{\lambda_2 t} \\ \frac{4}{3}e^{\lambda_1 t} - \frac{4}{3}e^{\lambda_2 t} & \frac{4}{3}e^{\lambda_1 t} - \frac{1}{3}e^{\lambda_2 t} \end{bmatrix}$$

$$\underline{y} = e^{\underline{A}t}\underline{c} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$