

SYSTEM OF ODES

consider the second order linear ODE : $y = y(t)$; $\frac{dy}{dt} = y'$

$$(1) \quad y'' + 5y' + 4y = r(x) \quad y(0) = 0, \quad y'(0) = 1$$

Homogeneous solution (when $r(x)$ is set to 0) should be determined first.

Homogeneous solution is the system behaviour when there is no input i.e., $r(x) = 0$.

Homogeneous solution :

$$y'' + 5y' + 4y = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \quad (\text{characteristic equation})$$

$$\Delta = 25 - 16 = 9$$

$$\Rightarrow \lambda_{1,2} = \frac{-5 \pm 3}{2} \begin{cases} -4 \\ -1 \end{cases}$$

So homogeneous solution

$$y_h(x) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-4t} + c_2 e^{-t}$$

(2) Now consider the first order linear ODE :

$$y' + 4y = r(t)$$

Homogeneous solution : $r(t) = 0$

$$y' + 4y = 0 \Rightarrow y' = -4y \Rightarrow \frac{dy}{dt} = -4y$$

$$\lambda + 4 = 0 \Rightarrow \lambda = -4$$

$$y_h = c e^{-4x}$$

$$\frac{dy}{y} = -4dt$$

$$\ln y = -4t + c'$$

$$y = c e^{-4t}$$

Now consider ODE (1) :

$$y'' + 5y' + 4y = 0 \quad (1)$$

Let $y_1 = y, y_2 = y', y_3 = y''$

$$y_1' = y' = y_2 \Rightarrow y_1' = y_2$$

$$y_2' = y'' = y_3 = -5y' - 4y = -5y_2 - 4y_1$$

From (1)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$

$$\frac{d}{dt} \underline{y} = \underline{A} \underline{y}$$

Then the solution

$$\underline{y} = e^{\underline{A}t} \underline{C}$$

How can we determine if $\underline{A}$ is a matrix?

Consider

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

Eigenvalues of $\underline{A}$:

Characteristic equation :

$$\det(\underline{A} - \lambda \underline{I}) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0$$

$$-\lambda(-5-\lambda) - (-4) = 0$$

$$\lambda^2 + 5\lambda + 4 = 0 \rightarrow \lambda_1 = -4, \lambda_2 = -1$$

Same characteristic equation as obtained before for the ODE.

$$\lambda_1 = -4, \lambda_2 = -1$$

Determine the corresponding eigen vectors

$$\underline{\lambda_1 = -4} :$$

$$(\underline{A} - \lambda \underline{I}) \underline{x}_1 = 0 \quad \underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$(\underline{A} + 4\underline{I}) \underline{x}_1 = 0 \Rightarrow \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$4x_1 + x_2 = 0$$

$$x_2 = -4, x_1 = 1 \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\text{Normalized: } \underline{x}_1 = \begin{bmatrix} 1/\sqrt{17} \\ -4/\sqrt{17} \end{bmatrix}$$

$$\lambda_2 = -1$$

$$(\underline{A} - \lambda \underline{I}) \underline{x}_2 = 0 \quad \underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

$$(\underline{A} + \underline{I}) \underline{x}_2 = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$$

$$x_1 + x_2 = 0$$

$$x_2 = -1, x_1 = 1 \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{Normalized: } \underline{x}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Now define

$$\underline{X} = [\underline{x}_1 \quad \underline{x}_2] = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}, \det(\underline{X}) = 3$$

$$\underline{X}^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix}$$

Remember:

$$\underline{D} = \underline{X}^{-1} \underline{A} \underline{X}$$

$$\underline{D} = \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 16 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4-16}{3} & 0 \\ 0 & \frac{-4+1}{3} \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that instead of $\underline{x}_1$ and $\underline{x}_2$; if we have used normalized eigenvectors (i.e., $\frac{\underline{x}_1}{\|\underline{x}_1\|}$ & $\frac{\underline{x}_2}{\|\underline{x}_2\|}$) the same result would be obtained. But then the property of $\underline{X}^{-1} = \underline{X}^T$ (orthogonality) would be satisfied. This property is used to simplify the quadratic form

$$Q = a x_1^2 + b x_1 x_2 + c x_2^2.$$

Then

$$\underline{Q} = \underline{X}^{-1} \underline{A} \underline{X} \Rightarrow \underline{X} \underline{Q} = \underbrace{\underline{X} \underline{X}^{-1}}_{\underline{I}} \underline{A} \underline{X} = \underline{I} \underline{A} \underline{X} = \underline{A} \underline{X}$$

$$\underline{A} \underline{X} = \underline{X} \underline{Q} \Rightarrow \underbrace{\underline{A} \underline{X} \underline{X}^{-1}}_{\underline{I}} = \underline{X} \underline{Q} \underline{X}^{-1} \Rightarrow \underline{A} = \underline{X} \underline{Q} \underline{X}^{-1}$$

$$\underline{A} = \underline{X} \underline{Q} \underline{X}^{-1}$$

$$\underline{A}^2 = \underline{A} \underline{A} = (\underline{X} \underline{Q} \underline{X}^{-1}) (\underline{X} \underline{Q} \underline{X}^{-1}) = \underline{X} \underline{Q}^2 \underline{X}^{-1}$$

$$\underline{Q} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \underline{Q}^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\underline{Q}^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

$$= \underline{X} \underline{X}^{-1} + \underline{X} \underline{D} \underline{X}^{-1} t + \frac{1}{2!} \underline{X} \underline{D}^2 \underline{X}^{-1} t^2 + \frac{1}{3!} \underline{X} \underline{D}^3 \underline{X}^{-1} t^3 + \dots$$

$$= \underline{X} \underline{X}^{-1} + \underline{X} \underline{D} \underline{X}^{-1} t + \underline{X} \frac{(\underline{D} t)^2}{2} \underline{X}^{-1} + \underline{X} \left[\frac{(\underline{D} t)^3}{3!} \right] \underline{X}^{-1} + \dots$$

$$= \underline{X} \left[I + \underline{D} t + \frac{(\underline{D} t)^2}{2!} + \frac{(\underline{D} t)^3}{3!} + \dots \right] \underline{X}^{-1}$$

$$= \underline{X} \begin{bmatrix} 1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \dots & 0 \\ 0 & 1 + \lambda_2 t + \frac{(\lambda_2 t)^2}{2!} + \dots \end{bmatrix} \underline{X}^{-1}$$

$$= \underline{X} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \underline{X}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} -1/3 & -1/3 \\ 4/3 & 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} e^{\lambda_1 t} & -\frac{1}{3} e^{\lambda_1 t} \\ \frac{4}{3} e^{\lambda_2 t} & \frac{1}{3} e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} e^{\lambda_1 t} + \frac{4}{3} e^{\lambda_2 t} & -\frac{1}{3} e^{\lambda_1 t} + \frac{1}{3} e^{\lambda_2 t} \\ \frac{4}{3} e^{\lambda_1 t} - \frac{4}{3} e^{\lambda_2 t} & \frac{4}{3} e^{\lambda_1 t} - \frac{1}{3} e^{\lambda_2 t} \end{bmatrix}$$

$$\underline{y} = e^{At} \underline{c} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y(a) \\ y'(a) \end{bmatrix}$$

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