Q1 A : Solve the following ODEs.

(i) $y' = -2y$

Sol'n: $\frac{dy}{dx} = -2y \Rightarrow \frac{dy}{y} = -2dx$

$\ln y = -2x + c^*$

$y = e^{-2x+c^*} = e^{-2x} \cdot \underbrace{e^{c^*}}_{c}$

$= ce^{-2x}$

Q2 (ii) $y' = 4e^{-x}\cos x$

Sol'n $\frac{dy}{dx} = 4e^{-x}\cos x \Rightarrow dy = 4e^{-x}\cos x\, dx$

$y = 4\int e^{-x}\cos x\, dx$

Use integration by parts methods to find the integral

$\int e^{-x}\cos x\, dx$

$\int u\,dv = uv - \int v\,du$

Let $u = e^{-x}$, $dv = \cos x\, dx \rightarrow v = \sin x$; $du = -e^{-x}dx$

$\int e^{-x}\cos x\, dx = e^{-x}\sin x + \int e^{-x}\sin x\, dx$ (1)

Similarly for $\int e^{-x}\sin x\, dx$

$u = e^{-x} \rightarrow du = -e^{-x}dx$; $dv = \sin x\, dx \rightarrow v = -\cos x$

Then

$\int e^{-x}\sin x\, dx = -e^{-x}\cos x - \int e^{-x}\cos x\, dx$ (2)

Substituting (2) into (1) :

$\int e^{-x}\cos x\, dx = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\, dx$

$2\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x) + c^*$

$\int e^{-x}\cos x\, dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + c'$ (3)

Verification :

$$e^{-x}\cos x = \frac{1}{2}\frac{d}{dx}e^{-x}(\sin x - \cos x)$$

$$= \frac{1}{2}\left[-e^{-x}(\sin x - \cos x) + e^{-x}(\cos x + \sin x)\right]$$

$$= e^{-x}\cos x \quad \checkmark$$

$$y = 4\int e^{-x}\cos x\, dx = 4\left(\tfrac{1}{2}\right)e^{-x}(\sin x - \cos x)$$

$$= 2e^{-x}(\sin x - \cos x) + C$$

(Q3) iii. $y'' = -y$

Sol'n : Assume $y = e^{\lambda x}$ is a solution.

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x}$$

$$y'' = -y \Rightarrow \lambda^2 \cancel{e^{\lambda x}} = -\cancel{e^{\lambda x}} \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \mp i$$

Then $y = c_1 e^{ix} + c_2 e^{-ix}$

$$= C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x)$$

$$= \underbrace{(C_1 + C_2)}_{A}\cos x + \underbrace{i(C_1 - C_2)}_{B}\sin x$$

$$y = A\cos x + B\sin x$$

is a solution. It satisfies the given ODE.

$$y' = -A\sin x + B\cos x$$

$$y'' = -A\cos x - B\sin x = -(A\cos x + B\sin x) -$$

$$y'' = -y \quad \checkmark$$

B Verification of a solution

(Q4) Show that y is a solution of the ODE. Determine the particular solution corresponding to the initial value given.

(1)

$$y' \tan x = 2y - 8 \quad \text{(ODE)}$$

$$y = c\sin^2 x + 4 \quad y\left(\tfrac{\pi}{2}\right) = 0$$

Sol'n $\quad y' = 2c\sin x \cos x$

$$y' \tan x = 2y - 8$$

$$\cancel{2c} \sin x \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}} = \cancel{2c}\sin^2 x + \cancel{8} - \cancel{8} \quad ✓$$

$$y = c\sin^2 x + 4$$

$$y\left(\tfrac{\pi}{2}\right) = 0 \quad 0 = c\sin^2\left(\tfrac{\pi}{2}\right) + 4 \Rightarrow c = -4$$

Therefore $y = -4\sin^2 x + 4$

C Separable ODE

(Q5) $$y' \sin \pi x = \pi y \cos \pi x$$

Sol'n $$\frac{dy}{dx} \sin \pi x = \pi y \cos \pi x$$

$$\frac{dy}{y} = \frac{\pi \cos \pi x}{\sin \pi x} dx$$

$$\int \frac{dy}{y} = \int \frac{\pi \cos \pi x}{\sin \pi x} dx \quad , \quad u = \sin \pi x, \quad du = \pi \cos \pi x \, dx$$

$$\ln y = \int \frac{du}{u} = \ln u$$

$$y = u + C = \sin \pi x + C$$

D. Reduction to Separable form $(u = y/x)$

(Q6) $xy' = y + 2x^3 \sin^2(\frac{y}{x})$

Sol'n $u = \frac{y}{x} \Rightarrow u' = \frac{y'x - y}{x^2} \Rightarrow xu' = \frac{y'x-y}{x} = y' - \frac{y}{x}$

$xu' = y' - u \Rightarrow \underline{y' = xu' + u}$

$xy' = y + 2x^3 \sin^2(\frac{y}{x})$

$y' = \frac{y}{x} + 2x^2 \sin^2(\frac{y}{x})$

$xu' + \cancel{u} = \cancel{u} + 2x^2 \sin^2 u$

$\cancel{x}u' = 2x^{\cancel{2}} \sin^2 u$

$\frac{du}{\sin^2 u} = \int 2x\,dx = x^2 + C$

$\int \frac{1}{\sin^2 u}\, du = \int \frac{1/\cos^2 u}{\sin^2 u/\cos^2 u}\, du = \int \frac{\sec^2 u}{\tan^2 u}\, du$

$w = \tan u \rightarrow dw = \sec^2 u\, du$

$\int \frac{1}{\sin^2 u}\, du = \int \frac{dw}{w^2} = -\frac{1}{w} = \frac{-1}{\tan u} = -\frac{1}{\tan(\frac{y}{x})}$

Then

$-\frac{1}{\tan(\frac{y}{x})} = x^2 + C$

$(x^2 + C)\tan(\frac{y}{x}) + 1 = 0$

E. Reduction to separable form

Q7. $y' = (4x+y+2)^2$ $\qquad u = 4x+y+2$

<u>Sol'n</u> $\qquad u' = 4 + y' \Rightarrow y' = u' - 4$

$$y' = (2x+y+2)^2$$
$$\downarrow$$
$$u' - 4 = u^2 \Rightarrow u' = u^2 + 4$$

$$\frac{du}{u^2+4} = dx \Rightarrow \int \frac{du}{u^2+4} = x + C$$

$$\rightarrow \int \frac{\frac{1}{4}du}{(\frac{u}{2})^2+1} = \int \frac{\frac{1}{2}dw}{w^2+1} \qquad w = \frac{u}{2},\ dw = \frac{1}{2}du$$

$$= \frac{1}{2}\tan^{-1} w = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4x+y+2}{2}\right)$$

Then

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+2}{2}\right) = x + C$$

F. Exact Differential Form (For differential continous function $\underline{u}$)

$$du = \underbrace{\frac{\partial u}{\partial x}}_{M(x,y)} dx + \underbrace{\frac{\partial u}{\partial y}}_{N(x,y)} \cdot dy = 0 \qquad (1)$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is satisfied, (1) is exact differential equation

Q8: Test for exactness. If exact solve.

$$\underbrace{(x^2+y^2)}_{M} dx + \underbrace{2xy}_{N} dy = 0$$

$$\left.\begin{aligned} M = x^2+y^2 &\rightarrow \frac{\partial M}{\partial y} = 2y \\ N = +2xy &\rightarrow \frac{\partial N}{\partial x} = 2y \end{aligned}\right\} \text{Equal} \Rightarrow \text{Exact}$$

$$M = \frac{\partial u}{\partial x} = x^2+y^2 \Rightarrow u = \int (x^2+y^2)\,\partial x = \frac{1}{3}x^3 + y^2 x + h(y)$$

$$N = \frac{\partial u}{\partial y} \Rightarrow 2xy = 2xy + h'(y) \rightarrow h'(y) = 0 \Rightarrow h'(y) = C_1$$

Then

$$u = c^*$$

$$\frac{1}{3}x^3 + y^2 x + C_1 = c^* \Rightarrow \frac{x^3}{3} + xy^2 = c$$

is a solution

G. Reduction to exact form

Q9 Test for exactness. If not use the given integrating factor.

$$\underbrace{e^{-y}}_{M} dx + \underbrace{e^{-x}(-e^{-y}+1)}_{N} dy = 0 \qquad F = e^{x+y} \qquad y(0)=1$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -e^{-y} \\ \frac{\partial N}{\partial x} &= -e^{-x}(-e^{-y}+1) \end{aligned}\right\} \text{ Not exact}$$

Multiply both sides by e^{x+y}:

$$e^{x}dx + e^{y}(e^{-y}+1)dy = 0$$

$$\underbrace{e^{x}}_{M}dx + \underbrace{(1+e^{y})}_{N}dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 0 \\ \frac{\partial N}{\partial x} &= 0 \end{aligned}\right\} \text{ Exact}$$

$$M = \frac{\partial u}{\partial x} = e^{x} \rightarrow \partial u = e^{x}\partial x$$

$$u = e^{x} + h(y) \qquad *$$

$$N = \frac{\partial u}{\partial y} \Rightarrow 1+e^{y} = h'(y) \rightarrow h(y) = y + e^{y} + c_1 \qquad **$$

$$u = c^*$$

$$e^{x} + y + e^{y} + c_1 = c^* \Rightarrow \underline{e^{x} + e^{y} + y = c}$$ is the solution

If $y(0)=1$ is given

$$e^{0} + e^{1} + 1 = c \Rightarrow c = 2 + e$$

Hence

$$e^{x} + e^{y} + y = 2 + e$$

is the solution.

H : First order Linear Differential Equations
(with constant coefficients)

Q10 $\quad y' + 3y = e^{-x} + x \qquad y(0) = -\frac{1}{9}$

Step 1 : Homogeneous solution.

$$y' + 3y = 0 \qquad y = e^{\lambda x}$$
$$\lambda + 3 = 0 \Rightarrow \lambda = -3$$
$$y_h = ce^{-3x}$$

step 2 : Particular solution for $e^{-x} + x$

Assume $\quad y_p = Ae^{-x} + K_1 x + K_0$

$$y_p' = -Ae^{-x} + K_1$$

$$y' + 3y = e^{-x} + x$$

$$-Ae^{-x} + K_1 + 3Ae^{-x} + 3K_1 x + 3K_0 = e^{-x} + x$$

$$\underbrace{(-A+3A)}_{1} e^{-x} + \underbrace{3K_1}_{1} x + \underbrace{K_1 + 3K_0}_{0} = e^{-x} + x$$

$$2A = 1 \rightarrow A = 1/2$$
$$K_1 = 1/3$$
$$K_1 + 3K_0 = 0 \Rightarrow K_0 = -\frac{1}{9}$$

Then $\quad y_p = \frac{1}{2} e^{-x} + \frac{1}{3} x - \frac{1}{9}$

Step 3 : Total solution :

$$y = y_h + y_p = ce^{-3x} + \frac{1}{2} e^{-x} + \frac{1}{3} x - \frac{1}{9}$$

$$y(0) = -\frac{1}{9} \Rightarrow \quad -\frac{1}{9} = c + \frac{1}{2} + 0 - \frac{1}{9}$$

$$-\frac{1}{9} - \frac{1}{2} + \frac{1}{9} = c \Rightarrow c = -\frac{1}{2}$$

$$y = -\frac{1}{2} e^{-3x} + \frac{1}{2} e^{-x} + \frac{1}{3} x - \frac{1}{9}$$

Q11 $\quad y' + 3y = e^{-3x} \qquad y(0) = 1$

Step 1 : Homogeneous solution (as before)

$$y_h = ce^{-3x}$$

Step 2 : Particular solution for e^{-3x}

$$y_p = Ae^{-3x}$$ × already exists in y_h.

$$y_p = Axe^{-3x}$$

$$y_p' = Ae^{-3x} - 3Axe^{-3x}$$

Substituting into ODE :

$$y' + 3y = e^{-3x}$$

$$Ae^{-3x} \cancel{- 3Axe^{-3x}} \cancel{+ 3Axe^{-3x}} = e^{-3x}$$

$$A = 1$$

$$y_p = xe^{-3x}$$

Step 3 : Total solution

$$y = y_h + y_p$$

$$y = ce^{-3x} + xe^{-3x}$$

$$y(0) = 1 \Rightarrow 1 = c$$

$$y = (1+x)e^{-3x}$$

Q12 $y' + 3y = 10\cos x \qquad y(0) = 4$

Step 1 : Homogeneous solution

$y' + 3y = 0$

$y_h = C e^{-3x}$ (as before)

Step 2 : Particular solution for $\cos x$

Assume $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y' + 3y = \cos x$

$\Downarrow$

$(-A\sin x + B\cos x) + 3(A\cos x + B\sin x) = 10\cos x$

$\underbrace{(B+3A)}_{=10}\cos x + \underbrace{(-A+3B)}_{=0}\sin x = 0$

$-A + 3B = 0 \Rightarrow A = 3B$

$B + 3A = B + 9B = 10 \Rightarrow B = 1,\ A = 3$

Then $y_p = 3\cos x + \sin x$

Step 3 : Total solution :

$y = Ce^{-3x} + 3\cos x + \sin x$

$y(0) = 4 \Rightarrow 4 = C + 3 \Rightarrow C = 1$

Hence $y = e^{-3x} + 3\cos x + \sin x$

General solution for second order Linear ODE with constant coefficients

Q13 $y'' + 5y' + 4y = 10e^{-3x}$ $y(0)=0,\ y'(0)=1$

Step 1 = Homogeneous solution

$y_h = e^{\lambda x}$

$\lambda^2 + 5\lambda + 4 = 0$

$\Delta = 25 - 16 = 9 \Rightarrow \lambda_{1,2} = \frac{-5 \mp 3}{2} \begin{cases} -4 \\ -1 \end{cases}$

$y_h = c_1 e^{-4x} + c_2 e^{-x}$

Step 2 = Particular solution for e^{-3x}

Assume $y_p = Ke^{-3x}$

$y_p' = -3Ke^{-3x}$

$y_p'' = 9Ke^{-3x}$

$y'' + 5y' + 4y = 10e^{-3x}$

$9K\cancel{e^{-3x}} - 15K\cancel{e^{-3x}} + 4K\cancel{e^{-3x}} = 10\cancel{e^{-3x}}$

$-2K = 10 \Rightarrow K = -5$

Then $y_p = -5e^{-3x}$

Step 3 : Total solution

$y = y_h + y_p$

$y = c_1 e^{-4x} + c_2 e^{-x} - 5e^{-3x}$

$y' = -4c_1 e^{-4x} - c_2 e^{-x} + 15e^{-3x}$

$y(0) = 0 \Rightarrow 0 = c_1 + c_2 - 5 \Rightarrow c_1 + c_2 = 5$

$y'(0) = 1 \Rightarrow 1 = -4c_1 - c_2 + 15 = 0$

$$\mp C_1 \mp C_2 = \mp 5$$
$$4C_1 + C_2 = 14$$
$$3C_1 = 9 \Rightarrow C_1 = 3$$
$$C_1 + C_2 = 5 \Rightarrow C_2 = 2$$

Then
$$y = 3e^{-4x} + 2e^{-x} - 5e^{-3x}$$

(Q14) $$y'' + 5y' + 4y = 9e^{-x} \qquad y(0) = 0,\ y'(0) = 0$$

Step 1 : Homogeneous solution
$$y'' + 5y' + 4y = 0$$
Solution is the same as y_h in Q13.
$$y_h = C_1 e^{-4x} + C_2 e^{-x}$$

step 2 : Particular solution for e^{-x} ;

Assume $y_p = Ke^{-x}$ (Not acceptable; such a term exists in y_h)
$$y_p = Kxe^{-x} \checkmark$$
$$y_p' = Ke^{-x} - Kxe^{-x}$$
$$y_p'' = -Ke^{-x} - K(e^{-x} - xe^{-x}) = -2Ke^{-x} + Kxe^{-x}$$

Substituting y_p, y_p' and y_p'' into the given differential equation.
$$y'' + 5y' + 4y = 9e^{-x}$$
$$\underbrace{-2K\cancel{e^{-x}} + Kx\cancel{e^{-x}}}_{y_p''} + 5\underbrace{(K\cancel{e^{-x}} - Kx\cancel{e^{-x}})}_{y_p'} + 4\underbrace{Kx\cancel{e^{-x}}}_{y_p} = 9$$
$$(-2K + 5K) + \underbrace{(K - 5K + 4K)}_{0} = 9$$
$$3K = 9 \Rightarrow K = 3$$

Then
$$y_p = 3xe^{-x}$$

Step 3 Total solution :

$$y = y_h + y_p$$

$$y = C_1 e^{-4x} + C_2 e^{-x} + 3x e^{-x}$$

$y(0) = 0 \Rightarrow 0 = C_1 + C_2 + 0$ $\boxed{C_1 + C_2 = 0}$ $C_1 = -C_2$

$$y' = -4C_1 e^{-4x} - C_2 e^{-x} + 3(e^{-x} - x e^{-x})$$

$y'(0) = 0 \Rightarrow 0 = -4C_1 - C_2 + 3 \rightarrow 4C_1 + C_2 = 3$

$3C_1 = 3 \Rightarrow C_1 = 1$

$C_2 = -1$

Then

$$y = e^{-4x} - e^{-x} + 3x e^{-x}$$

Q15 $y'' + 4y' + 8y = 4\cos x + 7\sin x$ $y(0) = 0,\ y'(0) = 0$

Step 1 = Homogeneous solution =

$$y'' + 4y' + 8y = 0$$

$\Delta = 16 - 32 = -16$ $\Rightarrow \lambda_1 = -2 + 2i;\ \lambda_2 = -2 - 2i$

Then

$$y_h = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x} = \alpha e^{-2x} e^{i2x} + \beta e^{-2x} e^{-i2x}$$

$$= e^{-2x}(\alpha \cos 2x + i\alpha \sin 2x + \beta \cos 2x - i\beta \sin 2x)$$

$$= e^{-2x}\Big[\underbrace{(\alpha+\beta)}_{C_1}\cos 2x + \underbrace{i(\alpha-\beta)}_{C_2}\sin 2x\Big]$$

Hence

$$y_h = e^{-2x}(C_1 \cos 2x + C_2 \sin 2x)$$

Step 2 = Particular solution for $r(x) = 4\cos x + 7\sin x$

$$y_p = A\cos x + B\sin x$$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x$$

Substituting y_p'', y_p' and y_p into the given differential equation.

$$y'' + 4y' + 8y = 4\cos x + 7\sin x$$

$$-A\cos x - B\sin x + 4(-A\sin x + B\cos x) + 8(A\cos x + B\sin x) = \cos x$$

$$(-A + 4B + 8A)\cos x + (-B - 4A + 8B)\sin x = \cos x$$

$$\underbrace{(7A+4B)}_{4}\cos x + \underbrace{(7B-4A)}_{7}\sin x = 4\cos x + 7\sin x$$

$$\left.\begin{aligned} 7A + 4B &= 4 \\ 7B - 4A &= 7 \end{aligned}\right\} A = 0,\ B = 1$$

Then

$$y_p = \sin x$$

Step 3: Total solution : $y = y_h + y_p$

$$y = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + \sin x$$

$$y(0) = 0 \Rightarrow e^{-2x}(c_1 + 0) + 0 = 0 \Rightarrow c_1 = 0$$

$$y = c_2 e^{-2x}\sin 2x + \sin x$$

$$y' = c_2(-2e^{-2x}\sin 2x + e^{-2x}\,2\cos x) + \cos x$$

$y'(0) = 0$.

$$0 = c_2(2) + 1 \Rightarrow c_2 = -1/2$$

Thefore

$$y = -0.5\,e^{-2x}\sin 2x + \sin x$$