## Laplace Equation

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, the both $u$ and $v$ satisfy the Laplace equation

$$\nabla^2 u = u_{xx} + u_{yy}$$

and

$$\nabla^2 v = v_{xx} + v_{yy}$$

$u_{xx}$ : second partial derivative $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

in D and have continuous second partial derivatives in D.

Proof : Cauch-Riemand equations :

$$\left.\begin{aligned} u_x &= v_y \quad \Rightarrow \quad u_{xx} = v_{yx} \\ u_y &= -v_x \quad \Rightarrow \quad u_{yy} = -v_{xy} \end{aligned}\right\}$$

Since $v_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = v_{xy}$

Then $u_{xx} + u_{yy} = 0$

Similarly we can derive

$$v_{xx} + v_{yy} = 0$$

The operator $\nabla^2$ is called Laplacian.

$$\nabla^2 f = f_{xx} + f_{yy}$$

Solutions of Laplacian equations are called harmonic functions. Hence the real and imaginary parts of an analytic function are harmonic functions.

If (i) $u$ and $v$ satisfy Cauch-Rieman equations

(ii) If $f = u + iv$ is an analyic function over D and

Then $v$ is said to be a Harmonic Conjugate function of $u$ over the domain D.

<u>Example:</u> $u = x^2 - y^2 - y$

(i) Show that $u$ is harmonic

(ii) Find the Harmonic Conjugate Function $v$ of $u$.

Solution:

(i) $\left.\begin{array}{l} u_x = 2x \rightarrow u_{xx} = 2 \\ u_y = -2y - 1 \rightarrow u_{yy} = -2 \end{array}\right\} \quad \nabla^2 u = 0$

(ii)

$$v_y = u_x = 2x \Rightarrow \frac{\partial v}{\partial y} = 2x \Rightarrow v = 2xy + h(x)$$

$$v_x = -u_y = 2y + h'(x) = -(-2y-1) = 2y + 1$$

$$\Rightarrow h'(x) = 1 \Rightarrow h(x) = x + C$$

Then $v = 2xy + x + C$

$$f(z) = u + iv = (x^2 - y^2 - y) + i(2xy + x + c)$$

$$= (x^2 - y^2 + i2xy) - y + ix + C$$

$$= (x+iy)^2 + i(x+iy) + C$$

$$= z^2 + iz + C$$

## 13.5 Exponential Function (630)

Complex exponential function:

$$f(z) = e^{z} = \exp(z)$$

$$e^{z} = e^{x+iy} = e^{x}(\cos y + i\sin y)$$

Properties:

(i) For $\cos y = 1,\ \sin y = 0 \Rightarrow e^{z} = e^{x}$ (Real)

(ii) $e^{z}$ is analyic for all $z$.

(iii) $(e^{z})' = \dfrac{de^{z}}{dz} = e^{z}$

(iv) $e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$

$$e^{z_1 z_2} = e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2)$$

$$= e^{x_1+x_2}\big[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\big]$$

$$= e^{x_1+x_2}\big[\cos(y_1+y_2) + i\sin(y_1+y_2)\big]$$

$$= e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}$$

Euler Formula: $e^{jy} = \cos y + i\sin y$

Periodicity of $e^{z}$:

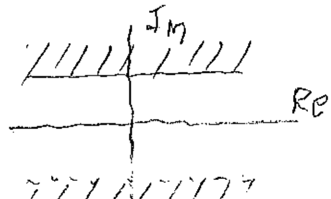
$$e^{z+i2n\pi} = e^{x+i(y\mp 2n\pi)}$$

$$= e^{x}\big[\cos(y\mp 2\pi n) + i\sin(y\mp 2\pi n)\big]$$

$$= e^{x}[\cos y + i\sin y] = e^{x+iy} = e^{z}$$

Periodic over $2\pi n$ strips over Im axis.

Fundamental region of $e^{z}$: $-\pi < y \le \pi$



Example:

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i\sin 0.6)$$

$$= 4.055(0.8253 - i\,0.5646).$$

Magnitude:

$$|e^{1.4-0.6i}| = |4.055(0.8253 - i\,0.5646)|$$

$$= |4.055|\underbrace{|0.8253 - i\,0.5646|}_{1} = 4.055$$

Argument

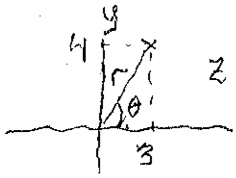
$$\operatorname{Arg}(e^{1.4-0.6i}) = -0.6$$

Example: $f(z) = e^{2+i}$ ; $g(z) = e^{4-i}$

Find $f(z)\cdot g(z) = (e^{2+i})(e^{4-i}) = e^{2+i+4-i} = e^{6}$

Example:

$$e^{z} = 3+4i = re^{iy} = e^{x}e^{iy}$$




$z = x+iy$

$$r = \sqrt{3^2+4^2} = \sqrt{9+16} = 5$$

$$\cos y = \frac{3}{5} = 0.6 \quad ; \quad \sin y = \frac{4}{5} = 0.8$$

$$e^{z} = e^{x}(\cos y + i\sin y)$$

$$\left.\begin{array}{l} e^{x} = 5 \Rightarrow x = \ln 5 = 1.609 \\ y = \cos^{-1} 0.6 = 0.927 \end{array}\right\} \begin{array}{l} z = x+iy \\ \quad = 1.609 + i\,0.927 \end{array}$$

In general $z = 1.609 + i(0.927 + 2n\pi)$

# 13.6. Trigonometric & Hyperbolic Functions (633)

Eulers formula

$$e^{iy} = \cos y + i\sin y$$

$$e^{-iy} = \cos y - i\sin y$$

Definition : Trigonometric functions

$$\cos z \triangleq \frac{1}{2}(e^{iz} + e^{-iz}) \;;\; (\cos z)' = \frac{1}{2}(ie^{iz} - ie^{iz}) = -\sin z$$

$$\sin z \triangleq \frac{1}{2i}(e^{iz} - e^{-iz}) \;;\; (\sin z)' = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \cos z$$

Then

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Definition : Hyperbolic Functions

$$\cosh z \triangleq \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z \triangleq \frac{1}{2}(e^{z} - e^{-z})$$

$$(\cosh z)' = \frac{1}{2}(e^{z} - e^{-z}) = \sinh z$$

$$(\sinh z)' = \frac{1}{2}(e^{z} + e^{-z}) = \cosh z$$

$$\tanh z = \frac{\sinh z}{\cosh z} \;;\; \coth z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \;;\; \operatorname{csch} z = \frac{1}{\sinh z}$$

## Real & Imaginary parts

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right]$$

$$= \frac{1}{2}\left[e^{ix-y} + e^{-ix+y}\right]$$

$$= \frac{1}{2}\left[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right]$$

$$= \frac{1}{2}\left[(e^{-y} + e^{y})\cos x + i(e^{-y} - e^{y})\sin x\right]$$

$$= \frac{1}{2}(e^{y} + e^{-y})\cos x - \frac{1}{2} i (e^{y} - e^{-y})\sin x \,]$$

$$= \cos x \cosh y - i \sin x \sinh z$$

Similarly

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Example

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Proof:

$$\cosh y = \frac{1}{2}(e^{z} + e^{-z}) \Rightarrow \cosh^2 y = \frac{1}{4}(e^{2z} + e^{-2z} + 2)$$

$$\sinh y = \frac{1}{2}(e^{z} - e^{-z}) \Rightarrow \sinh^2 y = \frac{1}{4}(e^{2z} + e^{-2z} - 2)$$

$$\cosh^2 y - \sinh^2 y = \frac{1}{4}(4) = 1 \Rightarrow \cosh^2 y = 1 + \sinh^2 y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$|\cos z|^2 = \cos^2 x \cdot \cosh^2 y + \sin^2 x \sinh^2 y$$

$$= \cos^2 x \cdot \cosh^2 y + (1 - \cos^2 x)\sinh^2 y$$

$$= \cos^2 x \underbrace{(\cosh^2 y - \sinh^2 y)}_{1} + \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y$$

Similarly

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Example

$$\cos z = 5$$

$$\frac{1}{2}(e^{iz}+e^{-iz}) = 5 \Rightarrow e^{iz}+e^{-iz} = 10$$

$$e^{iz} + \frac{1}{e^{iz}} = 10 \quad \Rightarrow \quad e^{iz} = t$$

$$t + \frac{1}{t} = 10 \Rightarrow t^2 + 1 = 10t \Rightarrow t^2 - 10t + 1 = 0$$

$$t_{1,2} = \frac{10 \mp \sqrt{100-4}}{2} = 5 \mp \sqrt{24} \begin{cases} 9.899 \\ 0.101 \end{cases}$$

$t_1 = 5 + \sqrt{24} = 9.899$

$t_2 = 5 - 24 = 0.101$

$t_2 = \frac{1}{t_1}$

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} \begin{cases} 9.899 \\ 0.101 \end{cases}$$

$$e^{-y} \begin{cases} t_2 \\ t_1 \end{cases} \Rightarrow e^{-y_1} = t_1 \Rightarrow e^{y_1} = \frac{1}{t_1} = t_1^{-1} \Rightarrow y_1 = \ln t_1^{-1}$$

$$y_1 = -\ln t_1 = -2.292$$

$$e^{-y_2} = t_2 = \frac{1}{t_1} \Rightarrow e^{y_2} = t_1 \Rightarrow y_2 = \ln t_1$$

$$y_2 = 2.292$$

$$x = 2\pi n$$

Then $z = \mp 2.292 + i2\pi n$

Relationship between trigonometric & hyperbolic functions

$$\cosh iz = \cos z \qquad \cos iz = \cosh z$$

$$\sinh iz = \sin z \qquad \sin iz = \sinh iz$$

# 13.7 Logarithm (636)

Complex natural logarithm

$$e^{w} = z \Rightarrow w = \ln z \qquad (z \neq 0)$$

$w = u + iv \;;\; z = x + iy$

$$e^{w} = z$$

$$e^{u+iv} = x + iy \Rightarrow e^{u}(\cos v + i \sin v) = x + iy$$

Then

$$x = e^{u} \cos v$$

$$y = e^{u} \sin v$$

or $\quad z = re^{i\theta} \Rightarrow e^{u+iv} = re^{i\theta}$

$$r = e^{u}, \; v = \theta$$

$$z = re^{i\theta} \Rightarrow \ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta}$$

$$= \ln r + i\theta$$

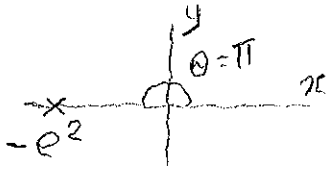
Indeed

$$z = re^{i(\theta \mp 2\pi n)} \Rightarrow \ln z = \ln r + i(\theta \mp 2\pi n)$$

<u>Example:</u> $z = e^{2} = e^{2} \cdot e^{\mp j 2\pi n}$

$$\ln z = \ln e^{2} + \ln e^{j 2\pi n} = 2 \mp j 2\pi n$$

<u>Example:</u> $z = -e^{2} = e^{2}\, e^{j(\pi \mp 2\pi n)}$



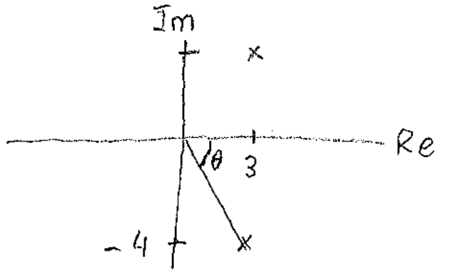

$$\ln z = \ln e^{2} + \ln e^{j\pi(1+2n)}$$

$$= 2 + j\pi(1 \mp 2n)$$

Multiplication & Division

$$\ln z_1 z_2 = \ln z_1 + \ln z_2 \;;\; \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

Example: $z = 3-4i \Rightarrow r = 5$




$$\theta = -\cos^{-1}\left(\frac{3}{5}\right) = -0.9273$$

$$\ln z = \ln\left(5e^{-i(0.9273 \mp 2\pi n)}\right)$$

$$= \ln 5 - i(0.9273 \mp 2\pi n)$$

Logarithmic function analytic

$$\ln z = \ln r + i(\theta \mp 2\pi n) = \underbrace{\ln r + i\theta}_{\mathrm{Ln}\, z \rightarrow \text{Principal value}} \mp i\,2\pi n$$

$$= \ln r + i(\theta + c)$$

$$= \ln\sqrt{x^2+y^2} + i\left(\tan^{-1}\frac{y}{x} + c\right)$$

$$= \underbrace{\frac{1}{2}\ln(x^2+y^2)}_{u} + \underbrace{i\left(\tan^{-1}\left(\frac{y}{x}\right) + c\right)}_{v}$$

$$u_x = \frac{\frac{1}{2}\,2x}{x^2+y^2} \quad ; \quad u_y = \frac{y}{x^2+y^2}$$

$$v_x = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} \quad ; \quad v_y = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}$$

$$(\ln z)' = u_x + i v_x = \frac{x}{x^2+y^2} + i\,\frac{-y/x^2}{1+(y/x)^2}$$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$$

<u>General Powers</u>

$$z^c = e^{\ln z^c} = e^{c\ln z}$$

$$\sqrt[n]{z} = z^{1/n} = e^{\ln z^{1/n}} = e^{\frac{1}{n}\ln z}$$

Example : $z = i^{i} = e^{i \ln i} =$

$$\ln i = \ln\left(e^{i\left(\frac{\pi}{2} \mp 2n\pi\right)}\right) = i\left(\frac{\pi}{2} \mp 2\pi n\right)$$

$$z = i^{i} = e^{i \ln i} = e^{i(i)\left(\frac{\pi}{2} \mp 2\pi n\right)} = e^{-\left(\frac{\pi}{2} \mp 2\pi n\right)}$$

Principal value : $z = e^{-\pi/2}$ $(n=0)$

Example :

$$z = (1+i)^{2-i} = e^{\ln (1+i)^{2-i}} = e^{(2-i)\ln(1+i)}$$

$$\ln(1+i) = \ln \sqrt{2}\, e^{i\left(\frac{\pi}{4} \mp 2n\pi\right)} = \ln\sqrt{2} + i\left(\frac{\pi}{4} \mp 2\pi n\right)$$

$$z = e^{(2-i)\left(\ln\sqrt{2} \mp i\left(\frac{\pi}{4} + 2\pi n\right)\right)}$$

$$= \exp\left[2\ln\sqrt{2} + \left(\frac{\pi}{4} \mp 2\pi n\right) + i\left(\frac{\pi}{2} + 4\pi n\right) - \ln\sqrt{2}\,\right]$$

$$= \exp\left[\ln\left(2 + \frac{\pi}{4} \mp 2\pi n\right) + i\left(\frac{\pi}{2} + 4\pi n - \frac{1}{2}\ln 2\right.\right]$$

$$= \left(2 + \frac{\pi}{4} \mp 2\pi n\right) + \left[\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right) + i\sin\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)\right.$$

$$z = 2 + \frac{\pi}{4} \mp 2\pi n + \left[\sin\left(\frac{1}{2}\ln 2\right) + i\cos\left(\frac{1}{2}\ln 2\right)\right]$$

Principal value : $n = 0$

$$z = 2 + \frac{\pi}{4} + \left[\sin\left(\frac{1}{2}\ln 2\right) + i\cos\left(\frac{1}{2}\ln 2\right)\right]$$

## Complex Exponentials

$$f(z) = a^z = e^{\ln a^z} = e^{z \ln a}$$

Example:

$$(1-i)^z = e^{z \ln(1+i)}$$

$$\ln(1-i) = \ln\left(\sqrt{2}\, e^{-\frac{\pi}{4} i}\right) = \ln\sqrt{2} + \ln e^{(-\frac{\pi}{4} i)}$$

$$= \frac{1}{2}\ln 2 - i\frac{\pi}{4}$$

$$e^{z \ln(1+i)} = \exp\left[(x+iy)\left(\frac{1}{2}\ln 2 - i\frac{\pi}{4}\right)\right]$$

$$= \exp\left[\left(\frac{1}{2}x\ln 2 + \frac{\pi}{4}y\right) + i\left(\frac{y}{2}\ln 2 - x\frac{\pi}{4}\right)\right.$$

$$= \exp\left[\frac{\ln 2}{2}(x+iy) + \frac{\pi}{4}(y - ix)\right]$$

$$= \exp\left[\frac{\ln 2}{2}(x+iy) - \frac{\pi}{4}i(x+iy)\right]$$

$$= \exp\left[\frac{\ln 2}{2}z - \frac{\pi}{4}iz\right].$$