14. COMPLEX INTEGRATION (643)

14.1. Line Integral (643)

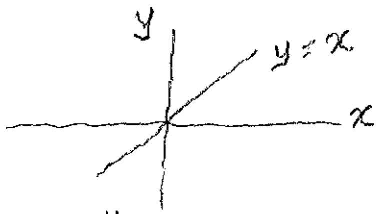
complex definite integrals are called complex line integral

$$\int_C f(z)\,dz$$

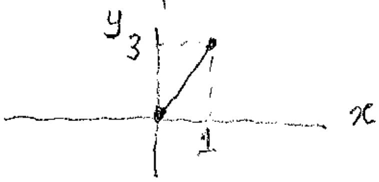
$f(z)$: The integrand

C : Given curve. C is represented by a parametric representation

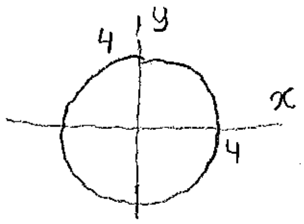
$$z(t) = x(t)$$



$z \ni x = y$: $\quad z(t) = t + it$



$z = z + 3it \quad (0 \le t \le 2)$ is a portion of the line $y = 3x$.



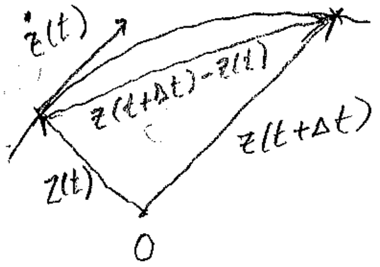
Circle with $|z| = 4$.

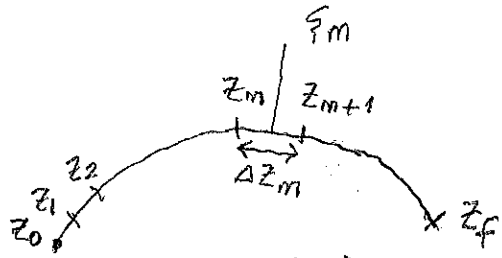
$z(t) = 4\cos t + 4i\sin t \quad (-\pi \le t \le \pi)$

Assume C be a smooth curve, ; C has a continuous and non-zero derivative

$$\dot{z}(t) = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$$

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$





$z_j = z(t_j)$

$z_0, z_1, z_2 \cdots \quad , z_n = z_f$

$$S_n = \sum_{m=1}^{n} f(\xi_m)\,\Delta z_m \quad ; \; \Delta z_m = z_m - z_{m-1}$$

In the limit as $\Delta z_m \to 0$, above sum is the definite integral over C.

If the curve C is closed, then

$$\oint_C f(z)\,dz$$

notation is used. ($z_i = z_f$).

Basic Properties

1. Linearity

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$$

2. Sense reversal

$$\int_{z_0}^{z_f} f(z)\,dz = -\int_{z_f}^{z_0} f(z)\,dz$$

3. Partitioning

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Existance

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m)$$
$$= \sum (u\Delta x_m - v\Delta y_m) + i\left[\sum u\Delta y_m + \sum v\Delta x_m\right]$$

So in the limit:

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz$$
$$= \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right]$$

First Evaluation Method

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

This formula is from calculus.

Theorem (Indefinite Integration of Analytic Function

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in a domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_f in D we have

$$\int_{z_0}^{z_f} f(z)\,dz = F(z_f) - F(z_0) \qquad [F'(z) = f(z)]$$

Example: $\int_0^{1+i} z^2\,dz = ?$

$$\int_0^{1+i} z^2\,dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{1}{3}(1+i)(1+i)^2$$
$$= \frac{1}{3}(1+i)(\cancel{1} - \cancel{1} + 2i) = \frac{1}{3}(2i - 2) = -\frac{2}{3} + \frac{2}{3}i$$

Example : $\int_{-\pi i}^{\pi i} \cos z \, dz = \sin z \Big|_{-\pi i}^{\pi i} = \sin \pi i - \sin(-\pi i)$

$$= 2\sin \pi i = \cancel{2}\,\frac{e^{i(2\pi i)} - e^{-i(2\pi i)}}{\cancel{2}i} \quad \left(\sin z = \frac{e^{iz}-e^{-iz}}{2}\right)$$

$$= \frac{e^{-2\pi} - e^{2\pi}}{i} = 2i\left(\frac{e^{2\pi}-e^{-2\pi}}{2}\right) = 2i\sinh(2\pi)$$

Second Evalution Method

Theorem2 (Integration by the use of the Path)

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt \quad \left(\dot{z} = \frac{dz}{dt}\right)$$

$$\dot{z} = \dot{x} + i\dot{y} \longrightarrow \qquad \dot{x}\,dt = \frac{dx}{dt}\cdot dt = dx$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$

$$= \int_a^b (u+iv)(\dot{x}+i\dot{y})\,dt$$

$$= \int_a^b (u+iv)(dx+i\,dy)$$

$$= \int_a^b (u\,dx - v\,dy) + i\int_a^b (u\,dy + v\,dx)$$

Example :

$I = \oint_C \frac{1}{z}\,dz = ?$ $\qquad$ C : The unit circle, counterclockwise

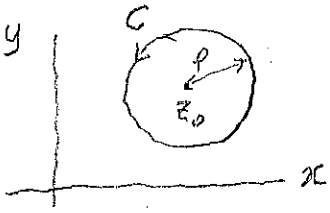
$$z = \cos t + i\sin t = e^{it} \qquad 0 \le t \le 2\pi$$

$$z(t) = e^{it} \Rightarrow dz = ie^{it}dt$$

$$I = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt = i\int_0^{2\pi} dt = i\,2\pi$$

Example:

$$\oint_C (z-z_0)^m dz = ?$$



$C: z(t) = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \quad (0 \leq t \leq 2\pi)$

$dz = \rho i e^{it} dt$

Then over C:

$$(z-z_0)^m = (\cancel{z_0} + \rho e^{it} - \cancel{z_0})^m = \rho^m e^{imt};$$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} dz$$

$$\frac{dz}{dt} = \rho i e^{it}$$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} \rho i e^{it} dt$$

$$= \rho^{m+1} \int_0^{2\pi} i\, e^{i(m+1)t} dt$$

$m = -1: (\rho^{m+1} = 1)$

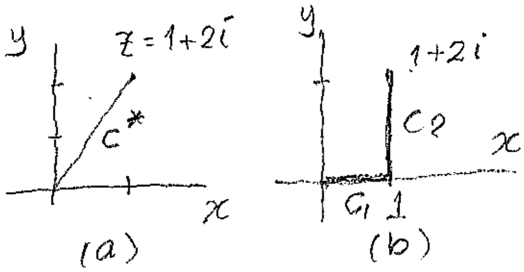
$$\oint_C (z-z_0)^m dz = 1 \int_0^{2\pi} i \cdot e^{i0} dt = i \int_0^{2\pi} dt = 2\pi i$$

$m \neq -1$

$$\oint_C (z-z_0) dz = \rho^{m+1} i \frac{1}{i(m+1)} e^{i(m+1)t} \Big|_0^{2\pi} = \frac{\rho^{m+1}}{m+1}(1-1) = 0$$

Example :

(a) $\displaystyle\int_{C^*} Re(z)\, dz$



On C^* $z = t + i2t \quad (0 \le t \le 1)$

$Re(z) = t \quad ; \quad dz = (1+i2)\,dt$

$$\int_{C^*} Re(z)\, dz = \int_0^{2\pi} (t)(1+i2)\, dt = \frac{t^2}{2}(1+i2)\Big|_0^1 = \frac{1}{2}(1+i2)$$

(b) $\displaystyle\int_C Re(z)dz = \int_{C_1} Re(z)dz + \int_{C_2} Re(z)dz$

$C_1 : z(t) = t \quad (0 \le t \le 1) \; ; Re(z) = t, \; dz = dt$

$C_2 : z(t) = 1 + it \; (0 \le t \le 2) \; ; Re(z) = 1 \; ; dz = i\,dt$

$$\int_C Re(z)\, dz = \int_0^1 t\, dt + \int_0^2 1\, i\,dt = \frac{t^2}{2}\Big|_0^1 + it\Big|_0^2 = \frac{1}{2} + i2$$

So integration is path dependent.

Bounds for Integrals : ML Inequality

$\displaystyle\int_C f(z)\, dz \le ML$ $\qquad$ L is the length of C

$\qquad |f(z)| \le M$ everywhere on C.

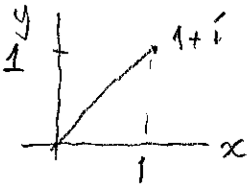
Proof :

$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m) \Delta z_m \right| \le \sum_{m=1}^{n} |f(\zeta_m)| \, |\Delta z_m| \le M \underbrace{\sum_{m=1}^{n} |\Delta z_m|}_{L}$$

$$\le ML$$

<u>Example</u>: Find an upper bound for the absolute value of the integral

$$\int_C z^2 dz$$



$$\left|\int_C z^2 dz\right| \leq ML \; ; \quad L = |1+i| = \sqrt{2}$$

$$|f(z)| = |z^2| = |z|^2 \leq (\sqrt{2})^2 = 2 \quad (\text{on } z)$$

$$\left|\int_C z^2 dz\right| \leqslant 2\sqrt{2}$$

$$\int_C z^2 dz = ? \qquad \text{On } C \; ; \; z = t + it \qquad 0 \leq t \leq 1$$

$$dz = (1+i)\,dt$$

$$\int_0^1 (t+it)^2 (1+i)\,dt = \int_0^1 t^2 (1+i)^3 dt = \frac{t^3}{3}(1+i)^3 \Big|_0^1 = \frac{1}{3}\sqrt{2}\, e^{i\frac{\pi}{4}3}$$

$$= \frac{1}{3}\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \frac{1}{3}(\sqrt{2})^3\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{3}(-1+i) = -\frac{2}{3} + \frac{2}{3}i$$

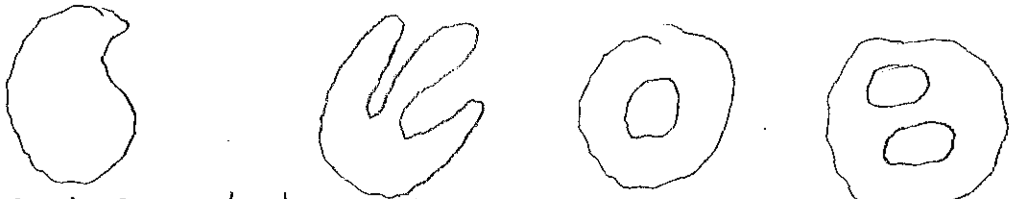
$$\left|\int_C z^2 dz\right| = \left|\frac{2}{3}(-1+i)\right| = \frac{2}{3}\sqrt{2}$$

14.2 Cauch's Integral Theorem (652)

A simple closed path

simple   simple   Not simple   Not simple

Simply connected domain



Simply connected   simply connected   Doubly connected   Triply connected

Cauchy's Integral Theorem

If $f(z)$ is analytic in simply connected D, then for every simply closed path C (also called a contour) in D

$$\oint_C f(z)\,dz = 0$$

The integral over a such a path a contour integral.

Example

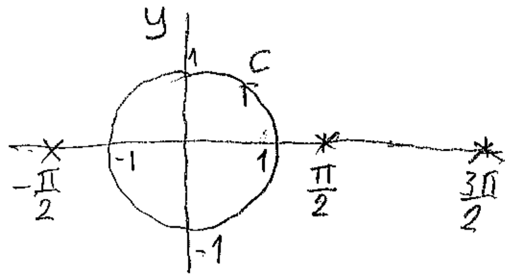
1. $\oint_C e^z\,dz = 0$

2. $\oint_C \cos z\,dz = 0$

3. $\oint_C z^n\,dz = 0$

Example :

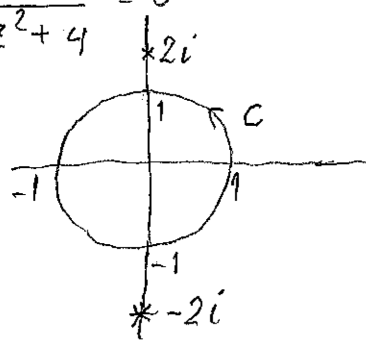
a) $\oint_C \sec z \, dz = 0$ C is unit circle.



$\sec z = \frac{1}{\cos z}$ is not analytic at $z = \mp \frac{\pi}{2}, \mp 3\frac{\pi}{2}$

These points are outside C.

b) $\oint \frac{dz}{z^2+4} = 0$

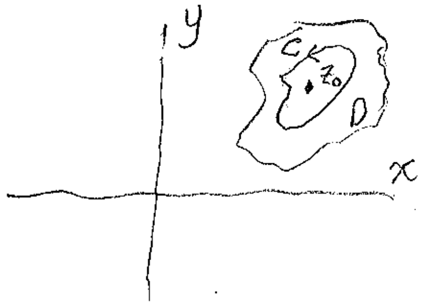


$\frac{1}{z^2+4}$ is not analytic at $z = \mp 2i$.

These points are outside c.

14.3 Cauchy's Integral Formula (660)

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and simply closed path C in D, that encloses z_0



$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

or

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Proof

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z_0)}{z-z_0}\,dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\,2\pi i$$

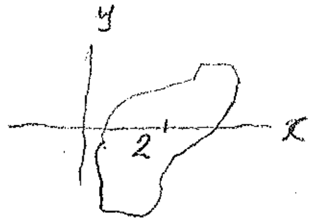
$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \underbrace{\frac{\varepsilon}{\rho}}_{=M}$$

The length of K is $\underbrace{2\pi\rho}_{L}$.

$$\left|\oint_C \frac{f(z)}{z-z_0}\,dz\right| \le \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon \to 0$$

Example : For any contour enclosing $z_0 = 2$

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i \underset{e^z}{\underset{\shortparallel}{f(z)}}\Big|_{z=2} = 2\pi i e^2$$



Example : For any contour $z = \frac{1}{2} i$

$$\oint \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{i}{2}}\,dz = 2\pi i \left[\frac{1}{2}z^3-3\right]_{z=\frac{i}{2}}$$

$$= 2\pi i\left[\frac{1}{2}\left(\frac{i}{2}\right)^3-3\right] = 2\pi i\left[\frac{1}{2}\left(-\frac{i}{8}\right)-3\right]$$

$$= \frac{\pi}{8} - 6\pi i$$

Example :

$$\oint_{C_i} \frac{z^2+1}{z^2-1}\,dz = ?$$

i. C_1 : A circle encloses $z_0 = 1$.

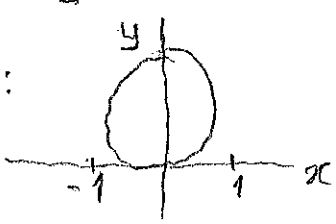
$$\oint_{C_1} \frac{z^2+1}{(z-1)(z+1)}\,dz = \int \frac{\overbrace{(z^2+1)/z+1}^{f(z)}}{z-1}\,dz = 2\pi i f(z_0=1)$$

$$= 2\pi i \left.\frac{z^2+1}{z+1}\right|_{z=1} = 2\pi i\,\frac{2}{2} = 2\pi i$$

ii. C_2 : A circle enclosing the point $z_0 = -1$

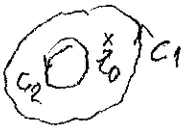
$$\oint_{C_2} \frac{\overbrace{(z^2+1)(z-1)}^{f(z)}}{(z+1)}\,dz = 2\pi i \left.\frac{z^2+1}{z-1}\right|_{z=-1} = 2\pi i\left(\frac{1+1}{-2}\right) = -2\pi i$$

iii. C_3 :



$$\oint_{C_3} \frac{z^2+1}{z^2-1}\,dz = 0 \;!$$

Example : Multiply connected domains



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

Example 1 : For any contour including $z_0 = 2$

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\, e^z\Big|_{z=2} = 2\pi i e^2$$

Example 2 : For any contour enclosing $z_0 = \frac{1}{2} i$

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{i}{2}}\,dz = 2\pi i\left(\frac{1}{2}z^3-3\right)\Big|_{z=\frac{i}{2}}$$

$$= 2\pi i\left(-\frac{i}{2\cdot 8}-3\right) = \frac{\pi}{8} - 6\pi i$$

14.4 Derivates of Analytic Function (664)

Theorem : If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic function in D. The values of these derivates at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

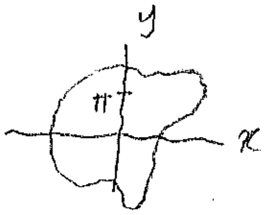
$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

or in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n=1,2,\ldots)$$

where C is any closed path D but encloses z_0 and whose full interior belongs to D, and we integrate counterclockwise around C.

Example For any contour enclosing the point πi (counterclockwise)



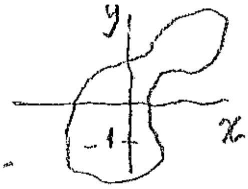
$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = ?$$

$$f(z) = \cos z \Rightarrow f'(z) = -\sin z \;;\; z_0 = \pi i$$

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i(\cos z)'\Big|_{z=\pi i} = -2\pi i \sin \pi i$$

$$= -2\pi \cancel{i}\,\frac{e^{i(\pi i)} - e^{-i(\pi i)}}{2\cancel{i}} = -2\pi\,\frac{e^{-\pi} - e^{\pi}}{2}$$

$$= 2\pi\,\frac{e^{\pi} - e^{-\pi}}{2} = 2\pi \sinh \pi$$



Example: For any contour enclosing the point $z_0 = -i$

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = ? \qquad f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}$$

$$f(z) = z^4 - 3z^2 + 6 \Rightarrow f'(z) = 4z^3 - 6z \Rightarrow f''(z) = 12z^2 - 6$$

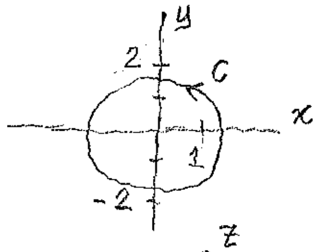
$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \frac{2\pi i}{2}(12z^2 - 6)\Big|_{z=-i} = \pi i\left[\,12(i)^2 - 6\,\right]$$

$$= \pi i(-12 - 6) = -18\pi i$$

Example :

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = ?$$

(a) For any contour C for which 1 lies and $\mp 2i$ lie outside.



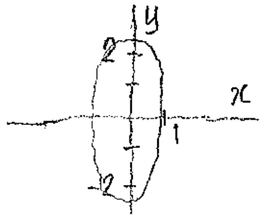
$$\oint_C \frac{e^z/(z^2+4)}{(z-1)^2}\,dz = 2\pi i\, f'(z_0) \quad (z_0 = 1)$$

$$f(z) = \frac{e^z}{z^2+4} \Rightarrow f'(z_0) = \left.\frac{e^z(z^2+4) - 2z e^z}{(z^2+4)^2}\right|_{z=1} = \frac{e(5)-2e}{(1+4)^2}$$

$$= \frac{3e}{25}$$

$$\oint_C \frac{e^z(z^2+4)}{(z-1)^2}\,dz = 2\pi i\,\frac{3e}{25} = \frac{6e\pi}{25}\,i$$

(b) For any contour C for which $\mp 2i$ lie inside and 1 lies outside



$$\oint_C \frac{e^z/(z-1)^2}{(z^2+4)}\,dz$$

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)} = \frac{A}{z+2i} + \frac{B}{z-2i} = \frac{(A+B)z + 2(A-B)i}{(z+2i)(z-2i)}$$

$$A + B = 0 \Rightarrow A = -B$$

$$2(2A)i = 1 \Rightarrow A = \frac{1}{4i}$$

$$\frac{1}{z^2+4} = \frac{1}{4i(z+2i)} - \frac{1}{4i(z-2i)}$$

$$\oint_C \left[\frac{1}{4i(z+2i)} - \frac{1}{4i(z-2i)}\right]\frac{e^z}{(z-1)^2}\,dz = \oint_C \frac{e^z\,dz}{4i(z+2i)(z-1)^2} - \oint_C \frac{e^z\,dz}{4i(z-2i)(z-1)^2}$$

$$= 2\pi i\left[\frac{e^z}{4i(z-1)^2}\right]_{z=-2i} - 2\pi i\left[\frac{e^z}{4i(z-1)^2}\right]_{z=2i}$$

$$= \frac{\pi}{2}\,\frac{e^{-2i}}{(-2i-1)^2} - \frac{\pi}{2}\,\frac{e^{2i}}{(2i-1)^2} = \frac{\pi}{2}\left[\frac{e^{-2i}}{(-3+4i)} + \frac{e^{2i}}{-3-4i}\right]$$

Cauchy's Inequality (pp 692)

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\int \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r$$

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$$

where C is a circle with radius "r" centered at $|z_0|$.

Liouville's Theorem :

If an entire function is bounded in absolute value in the whole complex plane, then the function must be a constant.

Morera's Theorem

If $f(z)$ is continuous in a simply domain D and if

$$\oint_C f(z)\,dz = 0$$

for every closed path in D, then $f(z)$ is analytic in D.