MATRIX ALGEBRA

7.1 Matrices (Definitions) (257)

$\underline{A}$ matrix is a rectangular array of numbers or functions which are enclosed in brackets.

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Rows

Columns

$m \times n$ matrix

$m \times n$; size of the matrix

Square matrix : $m = n$; an $n \times n$ matrix

Main diagonal for a square matrix : $a_{11}, a_{22}, \ldots a_{nn}$

Special cases :

A vector is a matrix with one row and column.

Row vector : $[a_1 \; a_2 \; \cdots \; a_n]$

Column vector : $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Equity :

$\underline{A} = [a_{jk}]$ and $\underline{B} = [b_{jk}]$ are equal ($\underline{A} = \underline{B}$) if and only if they have the same size and corresponding entries are equal, $a_{11} = b_{11}, a_{12} = b_{12}, \ldots, a_{mn} = b_{mn}$

Example :

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix} \Rightarrow$ $a_{11} = 2, a_{12} = 1, a_{13} = 3$
$a_{21} = 4, a_{22} = 0, a_{23} = -2$

Example :

$$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Addition :

$\underline{C} = \underline{A} + \underline{B}$: $\underline{A}$ & $\underline{B}$ are the same size. $C_{jk} = a_{jk} + b_{jk}$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 6 \\ 0 & 4 & 0 \end{bmatrix}$$

Scalar Multiplication

$$\alpha \underline{A} = \alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{33} \end{bmatrix}$$

$$-\underline{A} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{33} \end{bmatrix}$$

$$\underline{A} - \underline{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\underline{A} - \underline{B} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix}$$

$$0 \cdot \underline{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0} \rightarrow \text{zero matrix}$$

Rules :

$\underline{A} + \underline{B} = \underline{B} + \underline{A}$

$\underline{A} + (\underline{B} + \underline{C}) = (\underline{A} + \underline{B}) + \underline{C} = \underline{A} + \underline{B} + \underline{C}$

$\underline{A} + \underline{0} = \underline{A}$

$\underline{A} + (-\underline{A}) = \underline{0} \rightarrow$ Zero matrix

Other rules :

$$c(\underline{A}+\underline{B}) = c\,\underline{A} + c\,\underline{B}$$

$$(c+k)\underline{A} = c\,\underline{A} + k\,\underline{A}$$

$$c(k\underline{A}) = ck\,\underline{A}$$

$$1\cdot\underline{A} = \underline{A}$$

7.2. Matrix Muplication

Assume $\underline{A}$ is an $m\times n$ matrix, $\underline{B}$ is an $r\times p$ matrix, then $\underline{C} = \underline{A}\cdot\underline{B}$ is defined if and if $r=n$, the $r=n$ and $\underline{C}$ is an $m\times p$ matrix : $\underline{C} = [c_{jk}]$ with

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \dots + a_{jn}b_{nk} \qquad \begin{array}{l} j=1,2,\dots,m \\ k=1,2,\dots p \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & & c_{2m} \\ & & & \\ c_{m1} & c_{m2} & & c_{mp} \end{bmatrix}$$

# of columns

Example

$$\begin{array}{cc} & n=4 \\ m=3 & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 2 & 3 \\ 2 & 1 & 0 & 4 \end{bmatrix} \end{array} \begin{array}{c} p=3 \\ \begin{bmatrix} 4 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 3 & 2 \end{bmatrix} \leftarrow r=4 \end{array} = \qquad r=n\,!$$

$$\begin{bmatrix} 4+4+3+0 & 0+2+6+12 & 2+2+12+8 \\ 8+12+2+0 & 0+6+4+9 & 4+6+8+6 \\ 8+2+0+0 & 0+1+0+12 & 4+1+0+8 \end{bmatrix} = \begin{bmatrix} 11 & 20 & 34 \\ 32 & 19 & 24 \\ 10 & 13 & 13 \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \qquad \begin{aligned} \underline{a}_1 &= [1\ 2\ 3\ 4] \\ \underline{a}_2 &= [2\ 6\ 2\ 3] \\ \underline{a}_3 &= [2\ 1\ 0\ 4] \end{aligned}$$

$$\underline{B} = [\underline{b}_1\ \underline{b}_2\ \underline{b}_3] \ ; \ \underline{b}_1 = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} ; \ \underline{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} ; \ \underline{b}_3 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\underline{A}\,\underline{B} = \begin{bmatrix} \underline{a}_1\underline{b}_1 & \underline{a}_1\underline{b}_2 & \underline{a}_1\underline{b}_3 \\ \underline{a}_2\underline{b}_1 & \underline{a}_2\underline{b}_2 & \underline{a}_2\underline{b}_3 \\ \underline{a}_3\underline{b}_1 & \underline{a}_3\underline{b}_2 & \underline{a}_3\underline{b}_3 \end{bmatrix}$$

$$\underline{a}_1\underline{b}_1 = [4+4+3+0] = [11]$$

$$\underline{a}_1\underline{b}_2 = [0+2+6+12] = [20]$$

Example:

$$\begin{bmatrix} 6 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6+6 \\ 1+8 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8+2+3 \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 2 \\ 4 & 2 & 1 \\ 12 & 6 & 3 \end{bmatrix}$$

Example: Put the following set of equations in matrix form:

$$2x_1 + 3x_2 + x_3 = 4$$
$$x_1 + x_2 + 4x_3 = 2$$
$$3x_1 - x_2 - 2x_3 = -1$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

Important: Matrix multiplication is not commutative.

$$\underset{\sim}{A}\,\underset{\sim}{B} \neq \underset{\sim}{B}\,\underset{\sim}{A}$$

Consequently

$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{0}$ does not imply $\underset{\sim}{B}\,\underset{\sim}{A} = 0$

other rules:

$$k(\underset{\sim}{A})\underset{\sim}{B} = k(\underset{\sim}{A}\,\underset{\sim}{B}) = \underset{\sim}{A}(k\underset{\sim}{B})$$

$$\underset{\sim}{A}(\underset{\sim}{B}\,\underset{\sim}{C}) = (\underset{\sim}{A}\,\underset{\sim}{B})\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B}\,\underset{\sim}{C}$$

$$(\underset{\sim}{A}+\underset{\sim}{B})\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{C} + \underset{\sim}{B}\,\underset{\sim}{C}$$

$$\underset{\sim}{C}(\underset{\sim}{A}+\underset{\sim}{B}) = \underset{\sim}{C}\,\underset{\sim}{A} + \underset{\sim}{C}\,\underset{\sim}{B}$$

Example : $\underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x}$, $\underset{\sim}{x} = \underset{\sim}{B}\,\underset{\sim}{W}$ $\Big\}$ $\underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{B}\,\underset{\sim}{W}$

$\underset{\sim}{y} = \underset{\sim}{C}\,\underset{\sim}{W} \quad \Rightarrow \quad \underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B}$

Transposition

The transpose $\underset{\sim}{A}^T$ of a matrix $\underset{\sim}{A}$ is defined as

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (n \times m \text{ matrix})$$

$$\underset{\sim}{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \quad m \times n \text{ matrix !}$$

Example :

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 3 & 6 & 0 & 2 \\ 5 & -1 & -3 & 0 \end{bmatrix}^T = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 6 & -1 \\ 1 & 0 & -3 \\ 3 & 2 & 0 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 4 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

Example

$$\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & -1 \end{bmatrix}$$

Properties:

i- $(\underline{A}^T)^T = \underline{A}$

ii. $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$

iii. $(cA)^T = c\underline{A}^T$

iv. $(AB)^T = B^T A^T$

Special Matrices

Symmetric matrixes:

$$\underline{A}^T = \underline{A}$$

Skew-symmetric matrixes:

$$\underline{A}^T = -\underline{A}$$

Example:

$$\underline{A} = \begin{bmatrix} 12 & 9 & 7 \\ 9 & 4 & 5 \\ 7 & 5 & 2 \end{bmatrix} \rightarrow \text{symmetric}$$

Example:

$$\underline{A} = \begin{bmatrix} 0 & 9 & 7 \\ -9 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix} \rightarrow \text{skew-symmetric}$$

Triangular matrices

Upper triangulalar matrices:

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Diagonal

zero

Diagonal

all zero

Lower triangular matrices

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 5 \end{bmatrix}$$

→ Zero

→ Diagonal

Diagonal Matrices

$$\underset{\sim}{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Scalar matrix:

$$\underset{\sim}{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Unit matrix (Identity matrix) : I

$$\underset{\sim}{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: consider the production of two modules Alpha and Beta. The matrix $\underset{\sim}{A}$ shows the cost of each material([illegible])

$$\underset{\sim}{A} = \begin{array}{c l} \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} & \begin{array}{l} \rightarrow \text{Raw materials} \\ \rightarrow \text{Labor cost} \\ \rightarrow \text{other costs} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{Alpha} & \text{Beta} \end{array} & \end{array}$$

The matrix $\underset{\sim}{B}$ shows the production quantities for each quarter (number are in thousands].

$$\underset{\sim}{B} = \begin{array}{c l} \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} & \begin{array}{l} \rightarrow \text{Alpha} \\ \rightarrow \text{Beta} \end{array} \\ \begin{array}{cccc} Q1 & Q2 & Q3 & Q4 \end{array} & \end{array}$$

Determine the total cost in each quarter.

$$\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B} = \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2\times 3 + 1.6\times 6 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ & & & \end{bmatrix}$$

$$= \begin{array}{c l} \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} & \begin{array}{l} \rightarrow \text{Raw materials} \\ \rightarrow \text{Labour} \\ \rightarrow \text{other costs} \end{array} \quad \text{(in million TL)} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ Q1 & Q2 & Q3 & Q4 \end{array} & \end{array}$$

Example :

Assume $\underset{\sim}{A}$ gives the transition probabilites of a device operating properly or not properly (each year),

$$\begin{array}{cc} \text{Proper} & \text{Not proper} \end{array}$$

$$\underset{\sim}{A} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

Assume at the beginning 1000 devices are working properly, 100 devices working not properly.

What will be the status of the devices after first year

$$x = \begin{bmatrix} 1000 \\ 100 \end{bmatrix}$$

After first year :

$$y = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1000 \\ 100 \end{bmatrix} = \begin{bmatrix} 800 + 70 \\ 200 + 30 \end{bmatrix} = \begin{bmatrix} 870 \\ 230 \end{bmatrix} \begin{array}{l} \rightarrow \text{properly} \\ \rightarrow \text{not pr} \end{array}$$

After the second year :

$$z = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 870 \\ 230 \end{bmatrix} = \begin{bmatrix} 857 \\ 243 \end{bmatrix} \begin{array}{l} \rightarrow \text{properly} \\ \rightarrow \text{not properly} \end{array}$$

7.3 Linear System of Equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

m equations
n unknowns

If $b_1 = b_2 = \cdots = b_m = 0 \rightarrow$ Homogeneous system

If at least one $b_j \neq 0 \rightarrow$ Nonhomogeneous system.

Above set of linear equation can be written as

$$\underline{A}\,\underline{x} = \underline{b}$$

where

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

coefficient matrix
$m \times n$ matrix

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

n vector   m vector

Define a new matrix so called augmented matrix of the above system.

$$\tilde{\underline{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$(m+1) \times n$ matrix

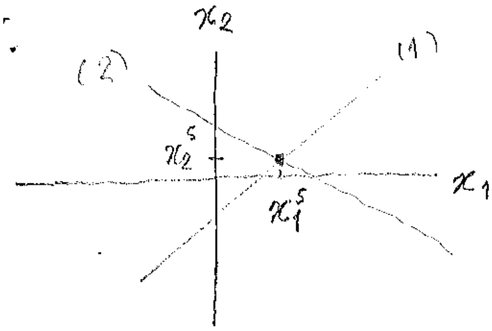
In order to determine x uniquely some conditions must be satisfied.

Example : Assume $m = n = 2$

$$a_{11} x_1 + a_{12} x_2 = b_1 \qquad (1)$$

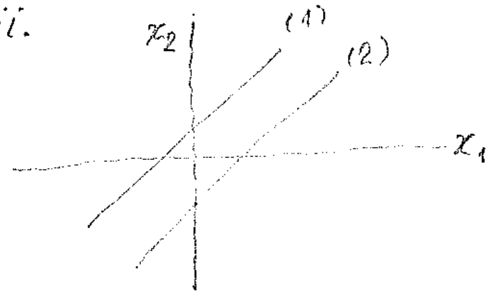
$$a_{21} x_1 + a_{22} x_2 = b_2 \qquad (2)$$

i.



unique solution : x_1^s, x_2^s

ii.



No solution

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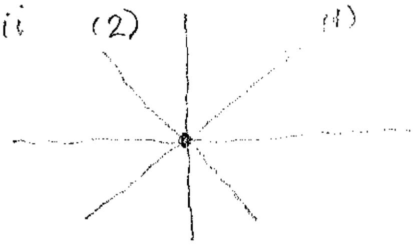

$b_1 = b_2$ (Homogeneous system)
Trivial solution $(0, 0)$

<u>Example :</u> Consider a linear system that is in triangular form.

$$2x_1 + 5x_2 = 12$$
$$3x_2 = 6$$

$$\begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$\Rightarrow 3x_2 = 6 \Rightarrow x_2 = 2$

$2x_1 + 10 = 12 \rightarrow 2x_1 = 2 \Rightarrow x_1 = 1$

Solution : $(x_1, x_2) = (1, 2)$

If the coeffient matrix is in triangular form, the solution can be obtained easily.

Gauss Elimination

Consider

$$2x_1 + 5x_2 = 12$$

$$x_1 + x_2 = 3$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 5 & 12 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow[\text{Row1} - \frac{1}{2}\text{Row2} \Rightarrow \text{Row 2}]{} \left[\begin{array}{cc|c} 2 & 5 & 12 \\ 0 & 3 & 6 \end{array}\right]$$

Then this system reduces to the previous example.

Further continue:

$$\left[\begin{array}{cc|c} 2 & 5 & 12 \\ 0 & 3 & 6 \end{array}\right] \xrightarrow[\frac{1}{3}\text{Row2} \to \text{Row 2}]{} \left[\begin{array}{cc|c} 2 & 5 & 12 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[\text{Row1} - 5\,\text{Row2} \to \text{Row1}]{} \left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow[\frac{1}{2}\text{Row1} \to \text{Row1}]{} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Then the modified system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<u>Note</u>

$$\underset{\sim}{I}\,\underset{\sim}{x} = \underset{\sim}{x}$$

$$\underset{\sim}{I}\,\underset{\sim}{A} = \underset{\sim}{A}$$

Example :

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \xrightarrow[\text{Row4} - 20\,\text{Row1} \Rightarrow \text{Row 4}]{\text{Row 1} + \text{Row 2} \Rightarrow \text{Row 2}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R3 - 3R2 \Rightarrow R3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-95x_3 = -190 \Rightarrow x_3 = 2$$

$$10x_2 + 25x_3 = 90 \Rightarrow 10x_2 + 50 = 90 \Rightarrow 10x_2 = 40 \Rightarrow x_2 = 4.$$

$$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$$

Example: (Infinitely Many Solutions)

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \xrightarrow[R3-0.4R1 \to R3]{R2-0.2R1 \to R2} \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \xrightarrow{R2+R3 \to R3} \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

x_1, x_2, x_3 & x_4 cannot be determined uniquely.

For example let $x_3 = 1, x_4 = 0$

$$\left.\begin{array}{l} 1.1x_2 + 1.1 = 1.1 \Rightarrow x_2 = 0 \\ 3x_1 + 0 + 2 - 0 = 8 \Rightarrow x_1 = 2 \end{array}\right\} \text{(2, 0, 1, 0) is a solution}$$

Let $x_3 = 0, x_4 = 1$

$$\left.\begin{array}{l} 1.1x_2 + 0 - 4.4 = 1.1 \Rightarrow 1.1x_2 = 5.5 \\ \qquad x_2 = 5 \\ 3x_1 + 10 + 0 - 5 = 8 \Rightarrow 3x_1 = 3 \Rightarrow x_1 = 3 \end{array}\right\} \text{(3, 5, 0, 1) is also a solution}$$

Example (No solution exists)

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \xrightarrow[-2R1+R3 \to R3]{-\frac{2}{3}R1+R2 \to R2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{3R2 \to R2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-2R2+R3 \to R3} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{array}\right] \to ? \quad 0 = 12 \,!$$ implies no solution.

No solution that satisfy all the equations.