Row Echelon Form

After the Gauss elimination, the row echelon form of the augmented matrix will be

$$\underline{R} = \left[\begin{array}{cccccc|c} r_{11} & r_{12} & \cdots & & & r_{1n} & f_1 \\ & r_{22} & \cdots & & & r_{2n} & f_2 \\ & 0 & & \ddots & & \vdots & \vdots \\ & & & & r_{rr} & r_{rn} & f_r \\ & & & & & & f_{r+1} \\ & & & 0 & & & \vdots \\ & & & & & & f_m \end{array}\right] \qquad r \leq m \,;\, r_{11} \neq 0$$

$\underline{R}$: Row reduced coefficient matrix $\underline{R}$

"r" : Rank of $\underline{R}$ and also rank of $\underline{A}$

(a) No solution :

(i) $r < m$ and

(ii) at least one of the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ is not zero.

$\underline{R}\,\underline{x} = \underline{f}$ is inconsistent then

(b) Unique solution :

System is consistent and $r = n$

(c) Infinitely many solutions

$r < m$, $f_{r+1} = f_{r+2} = \cdots = f_m = 0$

7.4 Linear Independence (Rank of a Matrix)

Linear Independence and Dependence of Vectors

$\underset{\sim}{a}_i = [a_{i1} \; a_{i2} \cdots a_{in}]$ $\quad \underline{a}_1, \underline{a}_2, \cdots, \underline{a}_m \rightarrow$ $1 \times n$ vectors (row vectors)

$c_1 \underline{a}_1 + c_2 \underline{a}_2 + \cdots + c_m \underline{a}_m$ $\rightarrow$ Linear combination of the above vectors

Consider the equation

$$c_1 \underline{a}_1 + c_2 \underline{a}_2 + \cdots + c_m \underline{a}_m = \underline{0}$$

Trivial solution: all c_i's are zero.

Linear independent : We cannot find a set of non zero c_i's.

Linear dependent : At least one set of scalars $\{c_1, c_2, \cdots, c_m\}$

or we can express $\underline{a}_1$ in terms of the other $\underline{a}_i$'s.

$$\underline{a}_1 = k_2 \underline{a}_2 + k_3 \underline{a}_3 + \cdots + k_m \underline{a}_m \quad \text{where } k_j = -c_j / c_1 .$$

Example:

$$\underline{a}_1 = [3 \quad 0 \quad 2 \quad 2]$$
$$\underline{a}_2 = [-6 \quad 42 \quad 24 \quad 54]$$
$$\underline{a}_3 = [21 \quad -21 \quad 0 \quad -15]$$

$$\underline{b}_1 = 2\underline{a}_1 + \underline{a}_2 = [0 \quad 42 \quad 28 \quad 58]$$
$$\underline{b}_2 = 7\underline{a}_1 - \underline{a}_3 = [0 \quad 21 \quad 14 \quad 29]$$
$$\underline{b}_3 = \underline{b}_1 - 2\underline{b}_2 = [0 \quad 0 \quad 0 \quad 0] = \underline{0}$$

$$\underline{b}_1 - 2\underline{b}_2 = \underline{0}$$
$$(2\underline{a}_1 + \underline{a}_2) - 2(7\underline{a}_1 - \underline{a}_3) = \underline{0}$$
$$-12\underline{a}_1 + \underline{a}_2 + 2\underline{a}_3 = \underline{0} \quad \rightarrow \text{Linearly dependent}$$
$$12\underline{a}_1 = \underline{a}_2 + 2\underline{a}_3 \Rightarrow \underline{a}_1 = \frac{1}{12}\underline{a}_2 + \frac{1}{6}\underline{a}_3$$

Rank of a Matrix

Maximum # of linearly independent row vectors of $\underline{A}$ is the rank of the matrix $\underline{A}$.

Example: Conside the above set of rowvectors given in the example: Rank is 2. (2 independent row vectors

Row Equivalent Matrices

A matrix $\underline{A}_1$ row equivalent to a matrix $\underline{A}_2$ if $\underline{A}_1$ can be obtained from $\underline{A}_2$ by elementary row operations. We have infinitely many $\underline{A}_1$.

Example:

$$\underline{A}_2 = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$\underline{A}_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 21 & 14 & 29 \end{bmatrix}$$

$$\underline{A}_1' = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$$

Rank in terms Column Vectors

The rank r of a matrix $\underline{A}$ equals the number of linearly independent column vectors $\underline{A}$.

Then $\underline{A}$ and $\underline{A}^T$ have the same rank.

Example

$$\underline{A}^T = \underline{A}_1 = \begin{bmatrix} 3 & -6 & 21 \\ 0 & 42 & -21 \\ 2 & 24 & 0 \\ 2 & 54 & -15 \end{bmatrix} \Rightarrow \underline{A}_2 = \begin{bmatrix} 3 & -6 & 21 \\ 0 & 42 & -21 \\ 0 & 28 & -14 \\ 0 & 58 & -29 \end{bmatrix} \Rightarrow \underline{A}_2' = \begin{bmatrix} 3 & -6 & 21 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow \underline{A}_2'' = \begin{bmatrix} 3 & -6 & 21 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Rank}: r = 2$$

Linear Dependence

Consider p vector each having n components.
If $n < p$, then these vectors are linearly dependent.

Assume the matrix $\underline{A}$ has p vectors as row vectors having n components ($\underline{A}$ is a $n \times p$ matrix)
If $n < p$, the rank of $\underline{A}$: $r \leq n < p$.

Vector Space

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

$$\underline{a}_1 = [1 \ 0 \ \cdots \ 0]$$
$$\underline{a}_2 = [0 \ 1 \ \cdots \ 0]$$
$$\vdots$$
$$\underline{a}_n = [0 \ 0 \ \cdots \ 1]$$

The row vectors obtained by $\underline{a}_i$'s as

$$c_1 \underline{a}_1 + c_2 \underline{a}_2 + \cdots + c_n \underline{a}_n$$

is called the row space of $\underline{A}$.
Similary the span of the column vectors of $\underline{A}$ is called the column space of $\underline{A}$.

7.5 Solutions of Linear Systems

Consider the following linear system of "m" equations in "n" unknowns $x_1, x_2, \dots, x_n$.

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{21} x_2 + \cdots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}
$$

with

$$
\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}, \quad \underline{\tilde{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right]
$$

i. Existance

Above set of equations is consistent, that is, has solutions (unique or multiple), iff the coefficient matrix $\underline{A}$ and the augmented matrix $\underline{\tilde{A}}$ have the same rank.

ii. Uniqueness

Above set of equations has precisely one solution iff $r = n$.

iii. Infinitely many solutions if $r < n$.

iv. Not solution can be concluded using Gauss Elimination metod.

Example:

$$3x_1 + 2x_2 + 2x_3 = 13$$
$$2x_1 + x_2 + x_3 = 7$$
$$x_1 + 2x_2 - x_3 = 2$$

$$\underline{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \qquad \underline{\tilde{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 2 & 13 \\ 2 & 1 & 1 & 7 \\ 1 & 2 & -1 & 2 \end{array}\right]$$

$$\underline{\tilde{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 2 & 13 \\ 2 & 1 & 1 & 7 \\ 1 & 2 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 7 \\ 3 & 2 & 2 & 13 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 3 & -3 & -3 \\ 0 & 4 & -5 & -7 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 4 & -5 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x_3 = 3$$
$$x_2 - x_3 = -1 \Rightarrow x_2 - 3 = -1 \Rightarrow x_2 = 2$$
$$x_1 + 2x_2 - x_3 = 2 \Rightarrow x_1 + 4 - 3 = 2 \Rightarrow x_1 = 1$$

Unique solution! $\{1, 2, 3\}$

Example

$$x_1 + 2x_2 - x_3 = 2$$
$$2x_1 + x_2 + x_3 = 7$$
$$3x_1 + 2x_2 + x_3 = 10$$

$$\underline{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \qquad \underline{\tilde{A}} = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 7 \\ 3 & 2 & 1 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 3 & -3 & -3 \\ 0 & 4 & -4 & -4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Multiple solution exists.

$x_2 - x_3 = -1 \quad ; x_3 = 2 \quad x_2 = 1$
$x_1 + 2x_2 - x_3 = 2 \Rightarrow x_1 + 2 - 2 = 2 \Rightarrow x_1 = 2$ $\Big\}$ $\{2, 1, 2\}$ is a solution

$x_2 - x_3 = -1 \quad ; x_3 = 3 \Rightarrow x_2 = 2$
$x_1 + 2x_2 - x_3 = 2 \Rightarrow x_1 + 4 - 3 = 2 \Rightarrow x_1 = 1$ $\Big\}$ $\{1, 2, 3\}$ is also a solution

7.6 Second and Third Order Determinants

The determinant of a 2×2 matrix $\underline{A}$:

$$D = \det \underline{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant is a useful tool to determine the solutions of linear system of equations

Consider linear system of two equations in two unknowns:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Hence

$$\begin{array}{cc} & \begin{array}{cc} x_1\text{'s} & (x_2\text{'s}) \end{array} \\ \tilde{\underline{A}} = & \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \end{array}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$ (Replace the column with $\underline{b}$) ; $D = a_{11}a_{22} - a_{12}a_{21}$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - a_{21}b_1}{D}$$

provided that $D \neq 0$.

If $D = 0$, the we have multiple solutions.

<u>Proof</u> : Solve the above equations. (HW)

Example: $4x_1 + 3x_2 = 10$
$2x_1 + 5x_2 = 12$

$$x_1 = \frac{\begin{vmatrix} 10 & 3 \\ 12 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{50-36}{20-6} = \frac{14}{14} = 1$$

$$x_2 = \frac{\begin{vmatrix} 4 & 10 \\ 2 & 12 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{48-20}{14} = \frac{28}{14} = 2$$

Third order Determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

Linear System of Three Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}$$

Note:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

So first column or first row may be used. The same approach for other columns or rows.

Example:

$$x_1 + 2x_2 - x_3 = 2$$
$$2x_1 + x_2 + x_3 = 7$$
$$3x_1 + 2x_2 + 2x_3 = 13$$

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{vmatrix} = (1)(2-2) - 2(4+2) + (3)(2+1) = -12 + 9 = -3$$

$$D_1 = \begin{vmatrix} 2 & 2 & -1 \\ 7 & 1 & 1 \\ 13 & 2 & 2 \end{vmatrix} = 2(2-2) - 7(4+2) + 13(2+1) = -42 + 39 = -3$$

$$D_2 = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 7 & 1 \\ 3 & 13 & 2 \end{vmatrix} = (1)(14-13) - 2(4+13) + 3(2+7)$$
$$= 1 - 34 + 27 = -6$$

$$D_3 = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 7 \\ 3 & 2 & 13 \end{vmatrix} = (1)(13-14) - 2(26-4) + 3(14-2) =$$
$$= -1 - 44 + 36 = -9$$

$$x_1 = \frac{D_1}{D} = \frac{-3}{-3} = 1;\quad x_2 = \frac{-6}{-3} = 2,\quad x_3 = \frac{-9}{-3} = 3$$

7.7 Cramer's Rule :(293)

consider an $n\times n$ (square) matrix

$$D = \det \underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

(i) $n = 1 \Rightarrow D = a_{11}$

(ii) $n \geq 2$

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}$$ (any row) $j=1,2,\cdots$ or n)

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk}$$ (any column) ($k=1,2\cdots$ or n)

$$C_{jk} = (-1)^{j+k} M_{jk}$$

where M_{jk} is a determinant of order $n-1$.

$$\therefore D = \sum_{k=1}^{n} a_{jk}(-1)^{j+k} M_{jk}$$ ($j = 1,2,\cdots$ or n : any row)

or

$$D = \sum_{j=1}^{n} a_{jk}(-1)^{j+k} M_{jk}$$ ($k = 1,2,\cdots$ or n : any column)

Example : $n=3$; $j=2$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \longrightarrow$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{23} & a_{33} \end{vmatrix} ; \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} ; \quad M_{31} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{21} = (-1)^3 M_{21} = -M_{21}$$
$$C_{22} = (-1)^4 M_{22} = M_{22}$$
$$C_{23} = (-1)^5 M_{23} = -M_{23}$$

$$D = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$$

Sign pattern:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example: Triangular Matrix

$$D = \begin{vmatrix} -3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 1 & 6 \end{vmatrix} = -3 \begin{vmatrix} 2 & 0 \\ 1 & 6 \end{vmatrix} = (-3)(12) = -36$$

Properties:

i. Interchange of two rows multiplies the value of the determinant by -1.

ii. Addition of a multiple of a row to another row does not alter the value of the determinant.

iii. Multiplication of a row by a nonzero constant c, multiplies the value of the determinant by c.

iv. An $n \times n$ matrix $\underline{A}$ has a rank n iff $\det \underline{A} \neq 0$.

v. Above statements are true for column operations.

Example: $\underset{\sim}{A} = \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$

i. $\det \underset{\sim}{A} = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = 16 - 12 = 4$

ii. $\det(\underset{\sim}{A}') = \begin{vmatrix} 4 & 6 \\ 0 & 1 \end{vmatrix} = 4$

iii $\det \underset{\sim}{A} = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = (2)(8-6) = 4$

$= (2)(2) \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = (4)(1) = 4$

iv. $\det(\underset{\sim}{A}) = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = (2)(2) = 4$

7.8 Inverse of a Matrix

$$\underline{A}\,\underline{A}^{-1} = \underline{A}^{-1}\underline{A} = \underline{I}$$

where

$$\underline{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \qquad (n \times n \text{ unit matrix})$$

If $\underline{A}$ has an inverse, then $\underline{A}$ is called a nonsingular matrix.

If $\underline{A}$ has no inverse, then $\underline{A}$ is called a singular matrix.

If $\underline{A}$ has an inverse, the inverse is unique.

The importantance of $\underline{A}^{-1}$:

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\underline{A}^{-1}\underline{A}\,\underline{x} = \underline{A}^{-1}\underline{b} \Rightarrow \underline{I}\,\underline{x} = \underline{A}^{-1}\underline{b}$$

$$\underline{x} = \underline{A}^{-1}\underline{b}$$

Gauss-Jordan Elimination

$$\underline{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

Not consider the augmented matrix:

$$[A \; I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1 + R_3 \to R3]{-2R_1 + R_2 \to R2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & -4 & 5 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{4}{3} R2 + R3 \to R3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right] \xrightarrow[R_3 + R_1 \to R_1]{-3R3 + R2 \to R2}$$

$$\sim \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2/3 & -4/3 & 1 \\ 0 & -3 & 0 & -1 & 5 & -3 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R2 \to R2}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2/3 & -4/3 & 1 \\ 0 & -1 & 0 & -1/3 & 5/3 & -1 \\ 0 & 0 & 1 & -1/3 & -4/3 & 1 \end{array}\right] \xrightarrow[-R2 \to R2]{2R2+R1 \to R1}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1/3 & -5/3 & 1 \\ 0 & 0 & 1 & -1/3 & -4/3 & 1 \end{array}\right]$$

$$\underbrace{}_{\underline{I}} \qquad \underbrace{}_{\underline{A}^{-1}}$$

Verification

$$\underline{A}\,\underline{A}^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 1/3 & -5/3 & 1 \\ -1/3 & -4/3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3} & 2-\frac{10}{3}+\frac{4}{3} & -1+2-1 \\ \frac{1}{3}-\frac{1}{3} & 4-\frac{5}{3}-\frac{4}{3} & \cdots \\ +\frac{2}{3}-\frac{2}{3} & \cdots & -3+2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$