Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ matrix $\underline{A}$ is given

$$\underline{A}^{-1} = \frac{1}{\det \underline{A}} [C_{jk}]^T = \frac{1}{\det \underline{A}} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\det \underline{A} = (1)(2-2) - 2(4-3) + (-1)(4-3) = 0 - 2 - 1 = -3$$

a_{11}	a_{12}	a_{13}
$C_{11} = 2-2 = 0$	$C_{12} = -(4-3) = -1$	$C_{13} = 4-3 = 1$
a_{21}	a_{22}	a_{23}
$C_{21} = -(4-(-2)) = -6$	$C_{22} = 2-(-3) = 5$	$C_{23} = -(2-6) = +4$
a_{31}	a_{32}	a_{33}
$C_{31} = 2-(-1) = 3$	$C_{32} = -(1-(-3)) = -4$	$C_{33} = 1-4 = -3$

$$\underline{A}^{-1} = \frac{1}{(-3)} \begin{bmatrix} 0 & -6 & 3 \\ -1 & 5 & \\ 1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 1/3 & -5/3 & 4/3 \\ -1/3 & -4/3 & 1 \end{bmatrix} \qquad D = -3$$

$$\underline{M} = \begin{bmatrix} 0 & 1 & 1 \\ 6 & 5 & -4 \\ 3 & 3 & -3 \end{bmatrix} \rightarrow \underline{M}^T = \begin{bmatrix} 0 & 6 & 3 \\ 1 & 5 & 3 \\ 1 & -4 & -3 \end{bmatrix} \xrightarrow[\underline{C}]{\frac{M^T}{D}} \begin{bmatrix} 0 & -2 & -1 \\ -1/3 & -5/3 & -1 \\ -1/3 & 4/3 & 1 \end{bmatrix}$$

change sign
$C_{ij} = -C_{ij}$ $i+j$ odd
$= C_{ij}$ even

$$\longrightarrow \underbrace{\begin{bmatrix} 0 & 2 & -1 \\ +1/3 & -5/3 & 1 \\ -1/3 & -4/3 & 1 \end{bmatrix}}_{A^{-1}}$$

Inverse of a 2x2 Matrix ($\det \underline{A} \neq 0$)

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(\underline{A}) = a_{11} a_{22} - a_{12} a_{21}$$

$$C_{11} = a_{22} \qquad C_{12} = -a_{21}$$

$$C_{21} = -a_{12} \qquad C_{22} = a_{11}$$

$$\underline{A}^{-1} = \frac{1}{\det(\underline{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Verification:

$$\underline{A}\,\underline{A}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{1}{\det \underline{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} a_{11} a_{22} - a_{12} a_{21} & -a_{11} a_{12} + a_{11} a_{12} \\ a_{21} a_{22} - a_{21} a_{22} & -a_{12} a_{21} + a_{11} a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Develop a rule for the inverse of a 2x2 matrix with $\det(\underline{A}) \neq 0$

Diagonal Matrices

$\underline{A} = [a_{jk}]$ $a_{jk} = 0$ when $j \neq k$. have an inverse iff all $a_{jj} \neq 0$.

$$\underline{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\underline{A}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & & 0 \\ \vdots & & & \\ 0 & 0 & & 1/a_{nn} \end{bmatrix}$$

Properties

$$(\underline{A}\,\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1} \quad \text{(In reverse order)}$$

Proof

$$(\underline{A}\,\underline{B})(\underline{A}\,\underline{B})^{-1} = \underline{A}\,\underbrace{\underline{B}\,\underline{B}^{-1}}_{\underline{I}}\,\underline{A}^{-1} = \underline{A}\,\underline{A}^{-1} = \underline{I}$$

$$\underline{B}(\underline{A}\,\underline{B})^{-1} = \underbrace{\underline{B}\,\underline{B}^{-1}}_{\underline{I}}\,\underline{A}^{-1} = \underline{A}^{-1}$$

$$(\underline{A}\,\underline{B})(\underline{A}\,\underline{B})^{-1} = \underline{A}\,\underline{A}^{-1} = \underline{I}$$

Matrix Multiplication

i. $\underline{A}\underline{B} = \underline{0}$ does not imply $\underline{A} = \underline{0}$ or $\underline{B} = 0$, or $\underline{B}\underline{A} = 0$

$$\underline{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{A}\,\underline{B} = \begin{bmatrix} -1+1 & 1-1 \\ -2+2 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but } \underline{A} \neq \underline{0} \text{ or } \underline{B} \neq 0$$

$$\underline{B}\,\underline{A} = \begin{bmatrix} -1+2 & -1+2 \\ 1-2 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \neq \underline{0}$$

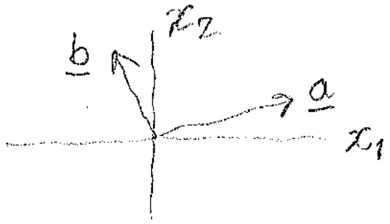
Determinant of a Matrix Product

$$\det(\underline{A}\underline{B}) = \det(\underline{A}) \cdot \det(\underline{B})$$

7.9 Vector spaces

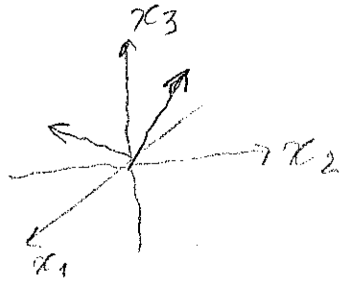
R^n : Real n-dimensional vector space R^n.

$n = 2$: vectors in the plane



$\underline{a}$ & $\underline{b}$ are column or row vectors.
They may be also matrices.

$n = 3$: vectors in 3D-space



Vector addition :

$$\underline{a}, \underline{b} \qquad \underline{a}+\underline{b} \quad , \underline{a} \in R^n, \underline{b} \in R^n$$

i. Commutativity

$$\underline{a}+\underline{b} = \underline{b}+\underline{a}$$

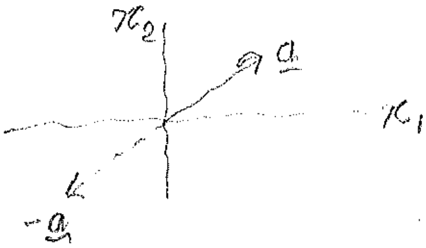
ii. Associativity :

$$(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$$

iii. Zero vector $\underline{0}$ (all components are zero)

$$\underline{a}+\underline{0} = \underline{a}$$

iv. The vector $-\underline{a}$



$$\underline{a}+(-\underline{a}) = \underline{0}$$

Scalar multiplication

c, k: scalar $\underline{a} \in R^n$, $\underline{b} \in R^n$

i. Distributivity:

$$c(\underline{a}+\underline{b}) = c\underline{a} + c\underline{b}$$

$$(c+k)\underline{a} = c\underline{a} + k\underline{a}$$

ii. Associativity

$$c(k\underline{a}) = (ck)\underline{a}$$

$$1\underline{a} = \underline{a}$$

Linear combination:

$$c_1\underline{a}_1 + c_2\underline{a}_2 + c_3\underline{a}_3 + \cdots + c_m\underline{a}_m$$

Linearly independent:

$$c_1\underline{a}_1 + c_2\underline{a}_2 + \cdots + c_m\underline{a}_m = 0$$

implies that $c_1 = c_2 = \cdots = c_m = 0$

If the vector space V has a dimension n or n-dimensional, then it contains a linearly independent set of n vectors.

$n = 2$: Two vectors

$n = 3$: Three vectors

Vector space of 2×2 matrices

Consider

$$\underline{B}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{B}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \underline{B}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \underline{B}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Any 2×2 matrix $\underline{A} = [a_{jk}]$ has a unique representation

$$\underline{A} = a_{11}\underline{B}_{11} + a_{12}\underline{B}_{12} + a_{21}\underline{B}_{21} + a_{22}\underline{B}_{22} = 0 \ ?$$

iff $a_{11} = a_{12} = a_{21} = a_{22}$. ; $\underline{B}_{11}, \underline{B}_{12}, \underline{B}_{21}, \underline{B}_{22}$ basis vectors

2D space : $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1} + b\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2}$ → column vectors

$\underline{B}_{11}, \underline{B}_{12}, \underline{B}_{21}, \underline{B}_{22}$: Basis vectors

$\underline{x}_1, \underline{x}_2$: Basis vectors

Inner Product Space

Assume $\underline{a}$ and $\underline{b}$ are column vectors.

$$\underline{a}^T\underline{b} = [a_1\ a_2\ \cdots\ a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= a_1b_1 + a_2b_2 + \cdots + a_nb_n \quad (\text{scalar})$$

$\underline{a}^T\underline{b} = (\underline{a}, \underline{b}) = \underline{a} \cdot \underline{b}$: Inner product (dot product)

Inner product space

i. $(q_1\underline{a} + q_2\underline{b}, \underline{c}) = (q_1\underline{a} + q_2\underline{b})^T c = q_1\underline{a}^T\underline{c} + q_2\underline{b}^T c$

$= q_1(\underline{a}, \underline{c}) + q_2(\underline{b}, \underline{c})$

ii. $(\underline{a}, \underline{b}) = \underline{a}^T\underline{b} = \underline{b}^T\underline{a}$

iii. $(\underline{a}, \underline{a}) \geq 0$

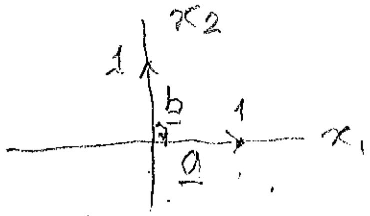
iii. $(\underline{a}, \underline{a}) = 0$ iff $\underline{a} = \underline{0}$

Orthogal vectors :

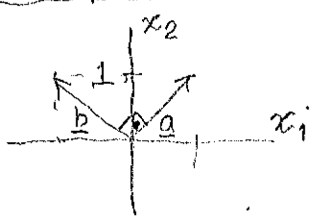
$$(\underline{a}, \underline{b}) = 0 \Rightarrow \underline{a} \,\&\, \underline{b} \text{ are orthogonal}$$

Example : $\underline{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$(\underline{a}, \underline{b}) = \underline{a}^T \underline{b} = [1 \; 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$



Example



$\underline{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; Determine $\underline{b}$ ∋ $\underline{a}$ and $\underline{b}$ are orthogonal.

$$(\underline{a}, \underline{b}) = 0$$

$a^T b = 0 \Rightarrow [1 \; 1] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \quad b_1 + b_2 = 0 \quad b_1 = 1 ; b_2 = -1$

$b_1 = -1 ; b_2 = 1$

Length or norm of a vector

$$\|\underline{a}\| = \sqrt{\underline{a}, \underline{a}} \geq 0$$

If the norm of vector is 1, then $\underline{a}$ is unit vector

Basic Results

i. $|(\underline{a}, \underline{b})| \leq \|a\| \, \|b\|$ (Cauchy-Schwarz inequality)

ii. $\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$ (Triangle Inequality)

iii. $\|\underline{a} + \underline{b}\|^2 + \|\underline{a} - \underline{b}\|^2 = 2(\|a\|^2 + \|b\|^2)$ (Parallelogram equality)

Linear Transformation (Linear Mapping)

$\underset{\sim}{v}$ & $\underset{\sim}{x} \in R^n$

$$\left.\begin{aligned} &F(\underset{\sim}{v}+\underset{\sim}{x}) = F(\underset{\sim}{v}) + F(\underset{\sim}{x}) \\ \& \quad &F(c\underset{\sim}{x}) = c\,F(\underset{\sim}{x}) \end{aligned}\right\}$$ Properties of linear transformation

Linear transformation :

$\underset{\sim}{x} \in R^n$, $\underset{\sim}{y} \in R^m$; $\underset{\sim}{A}$: An $n \times m$ matrix

$\underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x}$ $\Rightarrow$ Linear transformation

Example

$\underset{\sim}{x} \in R^n$, $\underset{\sim}{y} \in R^n$; $\underset{\sim}{A}$: $n \times n$ nonsingular matrix

$\underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x}$ $\Rightarrow$ Linear transform

$\underset{\sim}{A}^{-1}\underset{\sim}{y} = \underset{\sim}{A}^{-1}\underset{\sim}{A}\,\underset{\sim}{x} \Rightarrow \underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{y}$ (Inverse transform)

8
Matrix Eigenvalue Problems

8.1 The Eigenvalues and Eigenvectors of a Matrix A

Assume $\underset{\sim}{A}$ is an $n \times n$ nonzero matrix consider

$$\underset{\sim}{A}\,\underset{\sim}{x} = \lambda\,\underset{\sim}{x}$$

λ : Eigenvalues of $\underset{\sim}{A}$

$\underset{\sim}{x}$: Eigenvectors of A

Example:

$$\underset{\sim}{A} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}, \quad \underset{\sim}{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18+12 \\ 12+28 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

So $\lambda = 10$ is an eigenvalue

$\underset{\sim}{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector

Example

$$x = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix} = 3\begin{bmatrix} 11 \\ 9 \end{bmatrix}$$ No eigenvalue no eigenvector!

Theorem: An $n \times n$ matrix has at least one eigenvalue and atmost n numerically different eigenvalues

Definitions

$\underset{\sim}{A} - \lambda \underset{\sim}{I}$ = Characteristic matrix

$D(\lambda) = \det(A - \lambda I)$: Characteristic polynomial of A

$D(\lambda) = 0$: Characteristic equation of A

Solution : $\underline{A}\,\underline{x} = \lambda \underline{x} = \lambda \underline{I}\,\underline{x}$ $\quad \underline{I}$: $n\times n$ identity matrix

$$\underbrace{(\underline{A} - \lambda \underline{I})}_{\text{singular}}\underline{x} = 0$$

$$\det(\underline{A} - \lambda \underline{I}) = 0$$

Let $\underline{A} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}$, $\underline{A} - \lambda\underline{I} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 6-\lambda & 3 \\ 4 & 7-\lambda \end{bmatrix}$

$$\det(\underline{A} - \lambda\underline{I}) = (6-\lambda)(7-\lambda) - 12$$

$$= 42 - 13\lambda + \lambda^2 - 12 = \lambda^2 - 13\lambda + 30 = 0$$

$$\lambda_{1,2} = \frac{13 \mp \sqrt{169-120}}{2} = \frac{13 \mp 7}{2} \begin{cases} 10 \\ 3 \end{cases}$$

So there are two eigenvalues $\lambda_1 = 10$ & $\lambda_2 = 3$

Eigenvectors :

For $\lambda_1 = 10$:

$$(\underline{A} - 10\underline{I})\,\underline{x} = \underline{0}$$

$$\left(\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -4x_1 + 3x_2 &= 0 \\ 4x_1 - 3x_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} 4x_1 &= 3x_2 \\ x_2 = 4 &\to x_1 = 3 \end{aligned} \quad \Rightarrow \underline{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Check :

$$\underline{A}\,\underline{x} = \lambda \underline{x} \Rightarrow \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18+12 \\ 12+28 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \underset{\lambda}{10}\underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{x}$$

For $\lambda_2 = 3$

$$(\underline{A} - 3\underline{I})\,\underline{x} = 0 \Rightarrow \left(\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 3x_1 + 3x_2 &= 0 \\ 4x_1 + 4x_2 &= 0 \end{aligned}\right\} \quad x_1 + x_2 = 0 \Rightarrow x_2 = -x_1 \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Check :

$$\underline{A}\,\underline{x} = \lambda \underline{x} \Rightarrow \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \underset{\lambda}{3}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x}$$

Multiple Eigenvalues

Consider the following 3x3 matrix $\underline{A}$

$$\underline{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(1-\lambda)(-\lambda)-12\right] - 2(-2\lambda-6) - 3(-4+1-\lambda) = 0$$

$$= (-2-\lambda)(-\lambda+\lambda^2-12) - 2(-2\lambda-6) - 3(-3-\lambda) = 0$$

$$= -(2+\lambda)(\lambda^2-\lambda-12) + 4\lambda + 12 + 9 + 3\lambda = 0$$

$$= -2\lambda^2 + 2\lambda + 24 - \lambda^3 + \lambda^2 + 12\lambda + 7\lambda + 21 = 0$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45) = -(\lambda^3 - 125 + \lambda^2 - 25 - 21(\lambda-5) - 45 + 125 + 25 - 105$$

$$= -\left[(\lambda^3 - 5^3) + (\lambda^2 - 5^2) - 21(\lambda-5)\right] = 0$$

$$= -(\lambda-5)\left[(\lambda^2+5\lambda+25) + (\lambda+5) - 21\right] = 0$$

$$= -(\lambda-5)\left[\lambda^2+6\lambda+9\right] = -(\lambda-5)(\lambda+3)^2 = 0$$

$\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$

1. Eigenvector for $\lambda_1 = 5$

$$\underline{A} - 5\underline{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \xrightarrow[-7R_3+R_1 \rightarrow R_1]{\frac{7}{2}R_2+R_1 \rightarrow R_2} \begin{bmatrix} -7 & 2 & -3 \\ 0 & -12 & -24 \\ 0 & 16 & 32 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \cancel{-7} & \cancel{2} & \cancel{-3} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ Rank 2 !

$$\left.\begin{array}{l} x_2 + 2x_3 = 0 \rightarrow x_3 = -1,\ x_2 = 2 \\ -7x_1 + 2x_2 - 3x_3 = 0 \Rightarrow -7x_1 + 4 + 3 = 0 \Rightarrow x_1 = 1 \end{array}\right\} \underline{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\lambda_2 = \lambda_3 = -3$

$$A - \lambda I = \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

Generate two vectors :

$$\left.\begin{matrix} x_2 = 1 \\ x_3 = 0 \end{matrix}\right\} \quad x_1 + 2 = 0 \rightarrow x_1 = -2 \Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix}\right\} \quad x_1 + 0 - 3 = 0 \Rightarrow x_1 = 3 \Rightarrow x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Linearly independent!

Algebraic Multiplicity of λ :

The order of M_λ of an eigenvalue λ as a root.

- Above example

For $\lambda = 5$ → algebraic multiplicity is 1.

For $\lambda = -3$ → algebraic multiplicity is 2.

Geometric Multiplicity of λ :

The number of linearly independent eigenvectors to λ.

For $\lambda = 5$, geometric multiplicity is 1

$\lambda = -3$ " " is 2

Example :

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

$$\lambda_1 = \lambda_2 = 0$$

$\Rightarrow$ Algebraic multiplicity is 2. $(M_\lambda = 2)$

Eigenvectors : $\lambda = 0$

$$(\underline{A} - \lambda \underline{I})\, \underline{x} = 0 \Rightarrow \underline{A}\, \underline{x} = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0} \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = a \end{matrix}$$

Only one eigenvector ; $\rightarrow$ geometric multiplicity is 1. $(m_\lambda = 1)$

$$x = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

Defect of λ : $\Delta_\lambda = M_\lambda - m_\lambda = 1$!

Transpose :

The transpose $\underline{A}^T$ of a square matrix A has the same eigenvalues of $\underline{A}$