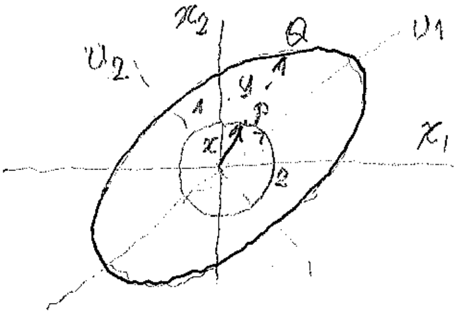
8.2 Some Examples of Eigenvalue Problems

Example: An elastic membrane in the x_1-x_2 plane with boundary $x_1^2 + x_2^2 = 1$. is streched so that a point $P:(x_1, x_2)$ goes over into the point $Q:(y_1, y_2)$ is given by



$$y_1 = 5x_1 + 3x_2$$
$$y_2 = 3x_1 + 5x_2$$

$$\underline{y} = \underline{A}\,\underline{x} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the principal directions, that is, the directions of the position vector $\underline{x}$ of P for which the direction of the position vector $\underline{y}$ of Q isinthe same or opposite direction.

$$\underline{y} = \lambda \underline{x} \Rightarrow \underline{A}\,\underline{x} = \lambda \underline{x} \quad \text{(Eigenvalue problem).}$$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \qquad (5-\lambda)^2 - 9 = 0$$

$$25 - 10\lambda + \lambda^2 - 9 = 0$$
$$\lambda^2 - 10\lambda + 16 = 0$$
$$(\lambda - 2)(\lambda - 8) = 0 \rightarrow \lambda_1 = 2, \lambda_2 = 8$$

Eigenvectors:

$$\lambda_1 = 2: \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \qquad 3x_1 + 3x_2 = 0 \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = 8: \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \qquad -3x_1 + 3x_2 = 0 \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

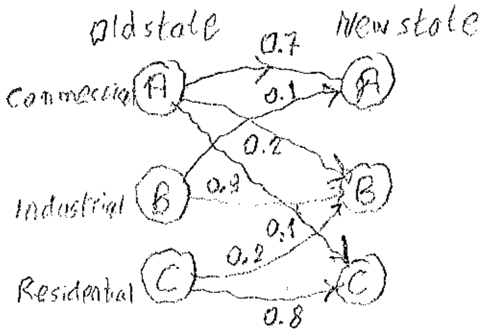
Rotation:

$$\underline{x} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \Rightarrow u_1 = 8\cos\phi \quad u_2 = 2\sin\phi$$

$$\cos^2\phi + \sin^2\phi = 1$$

$$\left(\frac{u_1}{8}\right)^2 + \left(\frac{u_2}{2}\right)^2 = 1 \rightarrow$$ Equation of the ellipse in (u_1, u_2) coordinates.

Example :



$$\begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$$

$\downarrow$ New $\qquad\qquad$ $\uparrow$ Old

What will be $\underline{x}$ at steady state so that $\underline{x}$ will stay constant

$$\underline{x} = \underline{A}\,\underline{x} \quad \text{or} \quad \underline{A}\,\underline{x} = \underline{x} \quad (\lambda = 1)$$

Eigenvector for $\lambda = 1$

$$(\underline{A} - \underset{1}{\underset{\shortparallel}{\lambda}} \underline{I})\underline{x} = \underline{0} \Longrightarrow \begin{bmatrix} 0.7-1 & 0.1 & 0 \\ 0.2 & 0.9-1 & 0.2 \\ 0.1 & 0 & 0.8-1 \end{bmatrix}$$

$$\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.2 & -0.7 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix} \longrightarrow \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0 & \frac{0.2}{0.3}0.1-0.7 & 0.2 \\ 0 & 0.1 & -0.6 \end{bmatrix} \rightarrow \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0 & -\frac{0.1}{3} & 0.2 \\ 0 & 0.1 & -0.6 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -0.3 & 0.1 & 0 \\ 0 & -0.1/3 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1/3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} -3x_A + x_B = 0 \\ -\frac{1}{3}x_B + 2x_C = 0 \end{array}$$

$$\text{Let } x_C = 1 \Rightarrow x_B = 6x_C = 6 \Rightarrow x_A = \frac{x_B}{3} = 2 \Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

So the installments in commercial-industrial-residential areas will approach to the ratio of 2 : 6 : 1

$$\text{For} \quad \underline{x} = \begin{bmatrix} 20 \\ 60 \\ 10 \end{bmatrix} \Rightarrow \underline{A}\,\underline{x} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 20 \\ 60 \\ 10 \end{bmatrix} = \begin{bmatrix} 14+6+0 \\ 4+54+2 \\ 2+0+8 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \\ 10 \end{bmatrix}$$

Assume initially $\underline{x}^{(0)} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$ Toplam hectar : 90

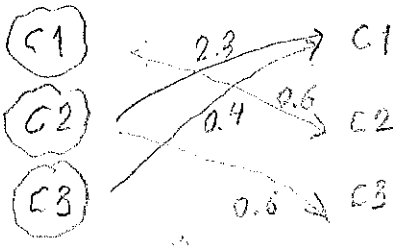
$$\underline{x}^{(1)} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = \begin{bmatrix} 21+3+0 \\ 6+27+6 \\ 3+24 \end{bmatrix} = \begin{bmatrix} 24 \\ 39 \\ 27 \end{bmatrix}$$

Toplam : 90 hectar

$$\underline{x}^{(2)} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 24 \\ 39 \\ 27 \end{bmatrix} = \begin{bmatrix} 20.7 \\ 45.3 \\ 24 \end{bmatrix} \rightarrow \begin{bmatrix} 19.02 \\ 49.71 \\ 21.27 \end{bmatrix} \rightarrow \cdots \begin{bmatrix} 18.15123 \\ 56.47053 \\ 15.37824 \end{bmatrix}$$

$$\cdots \rightarrow \begin{bmatrix} 20 \\ 60 \\ 10 \end{bmatrix}$$

Example : Class 1, Class 2, Class 3 A mammal speci
(Age 0-3) (Age 4-6) (Age 7-9)



$$\begin{bmatrix} C_1^{(n+1)} \\ C_2^{(n+1)} \\ C_3^{(n+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}}_{A} \begin{bmatrix} C_1^{(n)} \\ C_2^{(n)} \\ C_3^{(n)} \end{bmatrix} ; \quad \underline{C}^{(n+1)} = \underline{A}\,\underline{C}^{(n)}$$

Assume $\begin{bmatrix} C_1^{(0)} \\ C_2^{(0)} \\ C_3^{(0)} \end{bmatrix} = \begin{bmatrix} 6500 \\ 6500 \\ 6500 \end{bmatrix}$ Initially, total of 19500.

After 3 years :

$$\begin{bmatrix} C_1^{(1)} \\ C_2^{(1)} \\ C_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 2,3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 6500 \\ 6500 \\ 6500 \end{bmatrix} = \begin{bmatrix} 17550 \\ 3900 \\ 1950 \end{bmatrix}$$

After 6 years :

$$\begin{bmatrix} C_1^{(2)} \\ C_2^{(2)} \\ C_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 17550 \\ 3900 \\ 1950 \end{bmatrix} = \begin{bmatrix} 9750 \\ 10530 \\ 1170 \end{bmatrix}$$

After 9 years :

$$\begin{bmatrix} C_1^{(3)} \\ C_2^{(3)} \\ C_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \begin{bmatrix} 9750 \\ 10530 \\ 1170 \end{bmatrix} = \begin{bmatrix} 24687. \\ 5850 \\ 3159 \end{bmatrix}$$

Question : Is there a coefficient λ $\exists$

$$\underline{C}^{(n+1)} = \lambda\, \underline{C}^{(n)}$$

i.e., increase or decrease with a constant coefficient.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 2.3 & 0.4\\ 0.6 & -\lambda & 0\\ 0 & 0.3 & -\lambda\end{vmatrix}=-\lambda(\lambda^2-0)-0.6(-2.3\lambda-0.13)$$

$$=-\lambda^3+1.38\lambda+0.072=0$$

$\lambda=1.2$ is a root.

$$\begin{array}{l|l}
-\cancel{\lambda^3}+1.38\lambda+0.072 & \lambda-1.2 \\ \cline{2-2}
\underline{\pm\cancel{\lambda^3}\mp1.2\lambda^2} & -\lambda^2-1.2\lambda-0.06 \\
\quad -\cancel{1.2\lambda^2}+1.38\lambda+0.072 & \\
\quad \underline{\pm\cancel{1.2\lambda^2}\mp1.44\lambda} & \\
\qquad -0.06\lambda+0.072 & \\
\qquad \underline{\pm0.06\lambda\mp0.072} & \\
\qquad\qquad 0 &
\end{array}$$

Then

$$-\lambda^3+1.38\lambda+0.072=(\lambda-1.2)(-\lambda^2-1.2\lambda-0.06)$$

$$=-(\lambda-1.2)(\lambda^2+1.2\lambda+0.06)$$

$$\lambda_1=1.2\,;\ \lambda_{2,3}=\frac{-1.2\mp\sqrt{1.44-0.24}}{2}=\frac{-1.2\mp1.09}{2}\begin{cases}-1.145\\-0.055\end{cases}$$

Negative roots will not be meaningfull. Take $\lambda=1.2$

$$A-1.2I=\begin{bmatrix}-1.2 & 2.3 & 0.4\\ 0.6 & -1.2 & 0\\ 0 & 0.3 & -1.2\end{bmatrix}\Rightarrow\begin{aligned}-1.2c_1+2.3c_2+0.4c_3&=0\\0.6c_1-1.2c_2&=0\\0.3c_2-1.2c_3&=0\end{aligned}$$

Let $c_3=0.125\ \rightarrow\ 0.3c_2-0.15=0\Rightarrow c_2=0.5$

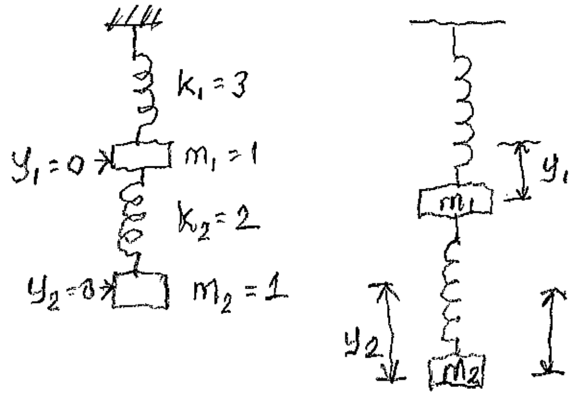
$$-1.2c_1+(2.3)(0.5)+(0.4)(0.125)=0\Rightarrow-1.2c_1+1.15+0.05=0\Rightarrow c_1=1$$

$\underset{\sim}{c}=\begin{bmatrix}1\\0.5\\0.125\end{bmatrix}$. To get a population of 1200, multiply by $\dfrac{19500}{1+0.5+0.125}=\dfrac{19500}{1.625}\approx12000$

$$\underset{\sim}{c}'=\begin{bmatrix}12000\\6000\\1500\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}0 & 2.3 & 0.4\\0.6 & 0 & 0\\0 & 0.3 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}12000\\6000\\1500\end{bmatrix}}_{\underset{\sim}{c}'}=\begin{bmatrix}14400\\7200\\1800\end{bmatrix}=\underset{\substack{\uparrow\\ \lambda}}{1.2}\underbrace{\begin{bmatrix}12000\\6000\\1500\end{bmatrix}}_{\underset{\sim}{c}'}$$

Example : Vibrating System of Two Masses on Two Springs



Note that net change in m_2 in length is $y_2 - y_1$.

It is given that

$$y_1'' = -3y_1 - 2(y_1 - y_2) = -5y_1 + 2y_2$$

$$y_2'' = \quad -2(y_2 - y_1) = +2y_1 - 2y_2$$

(" indicates second derivative)

$$\underline{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\underline{y}} =$$

Assume $\underline{y} = \underline{x} e^{\omega t} \Rightarrow \underline{y}' = \omega \underline{x} e^{\omega t} \Rightarrow \underline{y}'' = \omega^2 \underline{x} e^{\omega t}$

$$\underline{y}'' = \underline{A}\,\underline{y} \Rightarrow \omega^2 \underline{x} e^{\omega t} = \underline{A}\,\underline{x} e^{\omega t} \Rightarrow \underline{A}\,\underline{x} = \underset{\lambda}{\overset{}{\underset{\shortparallel}{\omega^2}}} \underline{x}$$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 4$$

$$= \lambda^2 + 7\lambda + 10 - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+1)(\lambda+6)$$

$\lambda_1 = -1$, $\lambda_2 = -6$

For $\lambda_1 = -1$: $\omega^2 = \lambda = -1 \Rightarrow \omega_{1,2} = \mp i$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} -4x_1 + 2x_2 = 0 \\ x_1 = 1 \rightarrow x_2 = 2 \Rightarrow \underline{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{matrix}$$

For $\lambda = -6 \rightarrow \omega^2 = -6 \Rightarrow \omega_{3,4} = \mp \sqrt{6}\, i$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + 2x_2 = 0 \\ x_1 = 2 \rightarrow x_2 = -1 \Rightarrow \underline{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{matrix}$$

Then there are four solutions :

$$\underline{y}^{(1)} = \underline{x}^{(1)} \cdot e^{it} \; ; \; \underline{y}^{(2)} = \underline{x}^{(1)} e^{-it} \; ; \; \underline{y}^{(3)} = \underline{x}^{(2)} e^{i\sqrt{6}t} \; ; \; \underline{y}^{(4)} = \underline{x}^{(2)} e^{-i\sqrt{6}t}$$

So the general solution :

$$\underline{y} = a_1 \underline{y}^{(1)} + a_2 \underline{y}^{(2)} + b_1 \underline{y}^{(3)} + b_2 \underline{y}^{(4)}$$

$$=$$

$$\underline{y} = \underline{x}^{(1)}\left[a_1 e^{it} + a_2 e^{-it}\right] + \underline{x}^{(2)}\left[b_1 e^{i\sqrt{6}t} + b_2 e^{-i\sqrt{6}t}\right]$$

$$= \underline{x}^{(1)}\left[\underbrace{(a_1+a_2)}_{a_1'}\cos t + \underbrace{i(a_1-a_2)}_{a_2'}\sin t\right] + \underline{x}^{(2)}\left[\underbrace{(b_1+b_2)}_{b_1'}\cos\sqrt{6}t + \underbrace{i(b_1-b_2)}_{b_2'}\sin\sqrt{6}t\right]$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}(a_1'\cos t + a_2'\sin t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}(b_1'\cos\sqrt{6}t + b_2'\sin\sqrt{6}t)$$

a_1', a_2', b_1' and b_2' coefficients are determined using initial conditions given.

For example : $y_1(0) = 0$, $y_2(0) = 0$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} a_1' + \begin{bmatrix} 2 \\ -1 \end{bmatrix} b_1' = \begin{bmatrix} a_1' + 2b_1' \\ 2a_1' - b_1' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Two more conditions must be given to determine all coeffients.

8.3 Symmetric, Skew Symmetric & Ortogonal Matrices

Symmetric Matrix :

$$\underline{A}^T = \underline{A}$$

Skew Symmetric Matrix :

$$\underline{A}^T = -\underline{A}$$

Orthogonal Matrix :

$$\underline{A}^T = \underline{A}^{-1} \Rightarrow \underline{A}\,\underline{A}^T = \underline{A}\,\underline{A}^{-1} = \underline{I}$$

Example

(i) $\underline{A} = \begin{bmatrix} 4 & 5 & -1 \\ 5 & 2 & -2 \\ -1 & -2 & 3 \end{bmatrix} \Rightarrow$ Symmetric : $\underline{A}^T = \begin{bmatrix} 4 & 5 & -1 \\ 5 & 2 & -2 \\ -1 & -2 & 3 \end{bmatrix} = \underline{A}$

(ii) $\underline{A} = \begin{bmatrix} 0 & 5 & -1 \\ -5 & 0 & -2 \\ +1 & 2 & 0 \end{bmatrix} \Rightarrow \underline{A}^T = \begin{bmatrix} 0 & -5 & 1 \\ 5 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} = -\underline{A}$ Skew symmetric !

(iii) $\underline{A} = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ $\underline{A}^T = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$

$$\underline{A} \,|\, \underline{I} = \left[\begin{array}{ccc|ccc} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -1 & -0.5 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} \frac{2}{3} & \frac{1}{3} & 0 & \frac{5}{9} & -\frac{2}{9} & \frac{4}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & -1.5 & -1 & -0.5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} \frac{2}{3} & 0 & 0 & \frac{4}{9} & -\frac{4}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & -1/5 & -1 & -0.5 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

Example:

$$\underline{A} = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}, \quad \underline{A}^T = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$\underline{A}\,\underline{A}^T = \begin{bmatrix} \frac{4+1+4}{9} & \frac{-4+2+2}{9} & \frac{2+2-4}{9} \\ \frac{-4+2+2}{9} & \frac{4+4+1}{9} & 0 \\ \frac{2+2-4}{9} & 0 & \frac{1+4+4}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I}$$

So $\underline{A}^T = \underline{A}^{-1}$ ⇒ Orthogonal matrix

Properties

Let $\underline{A}$ be a real matrix:

$\underline{R} = \frac{1}{2}[\underline{A} + \underline{A}^T]$ R is symmetric

$\underline{S} = \frac{1}{2}[\underline{A} - \underline{A}^T]$. S is skew symmetric

$\underline{R} + \underline{S} = \underline{A}$ ($\underline{A}$ can be decomposed into symmetric and skew symetric matrices.

Example

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \underline{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\underline{R} = \frac{1}{2}\begin{bmatrix} 2a_{11} & a_{12}+a_{21} \\ a_{21}+a_{12} & 2a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & \frac{1}{2}(a_{12}+a_{21}) \\ \frac{1}{2}(a_{12}+a_{21}) & a_{22} \end{bmatrix}$$ R is symmetric!

$$S = \frac{1}{2}\begin{bmatrix} 0 & a_{12}-a_{21} \\ a_{21}-a_{12} & 0 \end{bmatrix}$$ S is skew-symmetric!

Note that in a skew-symmetric matrix, diagonal entries are zero.

Example

$$\underline{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}, \quad \underline{A}^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

$$R = \frac{1}{2}(\underline{A} + \underline{A}^T) = \begin{bmatrix} 9 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$$

$$S = \frac{1}{2}(\underline{A} - \underline{A}^T) = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & -2 & 0 \end{bmatrix}$$

Theorem :

(a) The eigenvalues of a symmetric matrix are real.

(b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

The reverse are not true !

Orthoganal Transformations and Orthogonal Matrices

If $\underline{A}$ is an orthogonal matrix

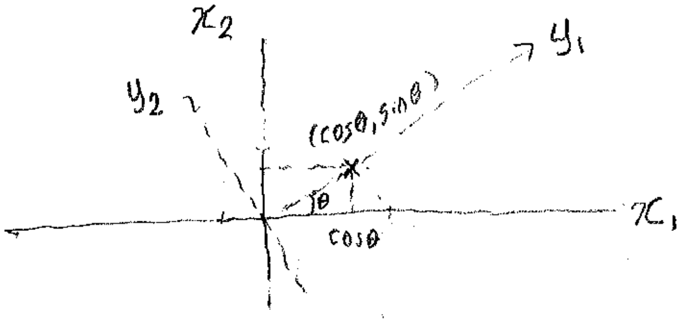
$$\underline{y} = \underline{A}\,\underline{x} \rightarrow \text{orthogonal transformation}$$

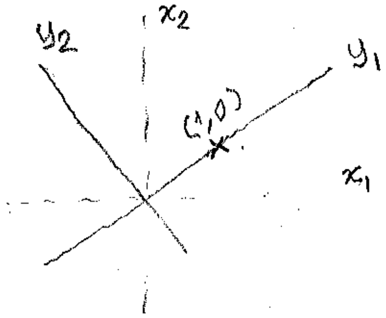
Exomple :

$$\underline{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \underline{A}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\underline{A}\,\underline{A}^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{A is an Orthogonal matrix}$$

$\underline{y} = A\,\underline{x}$ is an orthogonal transformation.

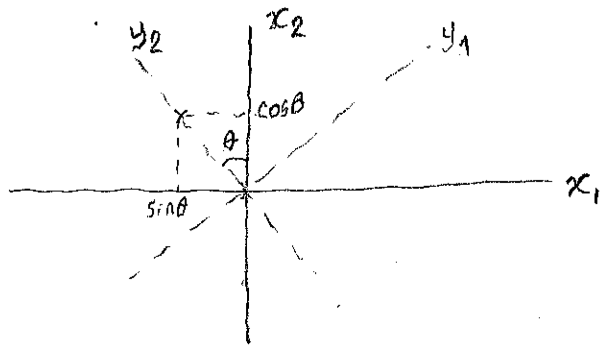


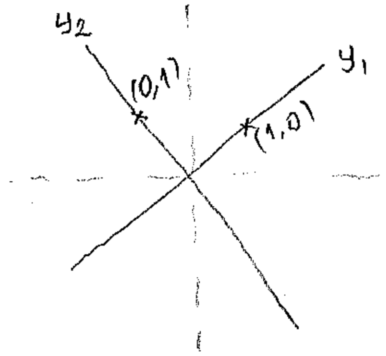


$$\underset{\sim}{x} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \quad , \quad \underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \quad \underline{y\ plane}$$

(x-plane)

$$\underset{\sim}{x} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} \qquad \underset{\sim}{y} = \underset{\sim}{A}\,\underset{\sim}{x} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$



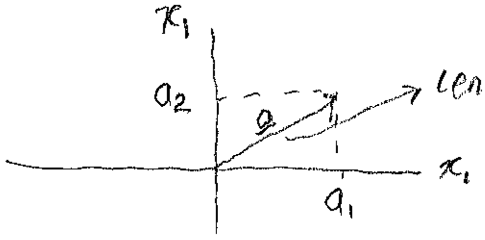


Inner Product

$$\underline{a}, \underline{b} \in R^n$$

$$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} = [a_1 \; a_2 \cdots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\|a\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{\underline{a}^T \underline{a}}$$ Norm or length of $\underline{a}$



length $= \sqrt{a_1^2 + a_2^2} = \sqrt{\underline{a}^T \cdot \underline{a}}$

$$\left.\begin{aligned} \underline{u} &= \underline{A}\,\underline{a} \\ \underline{v} &= \underline{A}\,\underline{b} \end{aligned}\right\}$$ orthogonal transformation of $\underline{a}$ and $\underline{b}$.

$$\underline{u} \cdot \underline{v} = (\underline{A}\,\underline{a})^T \cdot A\underline{b} = \underline{a}^T \underbrace{\underline{A}^T \underline{A}}_{\underline{I}} \underline{b} = \underline{a}^T \underline{b} = \underline{a} \cdot \underline{b}$$

Let $\underline{b} = \underline{a} \Rightarrow \underline{v} = \underline{u}$, then

$$\|\underline{u}\| = \|\underline{a}\|$$

Orthonormal System : Assume $\underset{\sim}{A}$ is an orthogonal matrix.

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \underset{\sim}{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a}_1^T \\ \underset{\sim}{a}_2^T \\ \vdots \\ \underset{\sim}{a}_n^T \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow \\ \underset{\sim}{a}_1 & \underset{\sim}{a}_2 & & \underset{\sim}{a}_n \end{matrix}$$

$$\underline{I} = \underset{\sim}{A}^{-1}\underset{\sim}{A} = \underset{\sim}{A}^T\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{a}_1^T \\ \underset{\sim}{a}_2^T \\ \vdots \\ \underset{\sim}{a}_n^T \end{bmatrix} [\underset{\sim}{a}_1\, \underset{\sim}{a}_2 \cdots \underset{\sim}{a}_n] = \underbrace{\begin{bmatrix} \overset{1}{\overset{\shortparallel}{\underset{\sim}{a}_1^T\underset{\sim}{a}_1}} & \overset{0}{\overset{\shortparallel}{\underset{\sim}{a}_1^T\underset{\sim}{a}_2}} & \cdots & \overset{0}{\overset{\shortparallel}{\underset{\sim}{a}_1^T\underset{\sim}{a}_n}} \\ \vdots & \ddots & & \\ \underset{\sim}{a}_n^T\underset{\sim}{a}_1 & \cdots & & a_n^T a_n \end{bmatrix}}_{I}$$

$$\underset{\sim}{a}_j^T\underset{\sim}{a}_k = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Theorem: $\det(A^T) = \det(\underset{\sim}{A})$

Theorem: The determinant of an orthogonal matrix has the value +1 or -1.

Proof:

$$\det(\underset{\sim}{A}\underset{\sim}{A}^{-1}) = \det(\underline{I}) = 1$$

$$\det(\underset{\sim}{A}\underset{\sim}{A}^T) = (\det\underset{\sim}{A})(\det\underset{\sim}{A}^T) = (\det\underset{\sim}{A})(\det\underset{\sim}{A}) = [\det(\underset{\sim}{A})]^2 = 1$$

$$\det(A) = \mp 1$$

Theorem : The eigenvalues of an orthogonal matrix $\underset{\sim}{A}$ are real or complex conjugates in pairs and have absolute value 1.

Example: $\underline{A} = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$

Characteristic Equation:

$$\det(\underline{A}-\lambda\underline{I}) = \begin{vmatrix} \frac{2}{3}-\lambda & 1/3 & 2/3 \\ -\frac{2}{3} & \frac{2}{3}-\lambda & 1/3 \\ \frac{1}{3} & 2/3 & -\frac{2}{3}-\lambda \end{vmatrix}$$

$$=\left(\frac{2}{3}-\lambda\right)\left[\left(\frac{2}{3}-\lambda\right)\left(-\frac{2}{3}-\lambda\right)-\frac{2}{9}\right]-\frac{1}{3}\left[-\frac{2}{3}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{9}\right]+\frac{2}{3}\left[-\frac{4}{9}-\frac{1}{3}\left(\frac{2}{3}-\lambda\right)\right]$$

$$=\left(\frac{2}{3}-\lambda\right)\left[\lambda^2-\frac{4}{9}-\frac{2}{9}\right]-\frac{1}{3}\left[+\frac{2}{3}\lambda+\frac{4}{9}-\frac{1}{3}\right]+\frac{2}{3}\left[+\frac{1}{3}\lambda-\frac{4}{9}-\frac{2}{9}\right]$$

$$=\left(\frac{2}{3}-\lambda\right)\left(\lambda^2-\frac{2}{3}\right)-\frac{1}{3}\left(+\frac{2}{3}\lambda+\frac{1}{3}\right)+\frac{2}{3}\left(+\frac{1}{3}\lambda-\frac{2}{3}\right)$$

$$=-\lambda^3+\frac{2}{3}\lambda^2+\underline{\frac{2}{3}\lambda}-\underline{\frac{4}{9}}-\underline{\frac{2}{9}\lambda}-\underline{\frac{1}{9}}+\underline{\frac{2}{9}\lambda}-\underline{\frac{4}{9}}$$

$$=-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda-1$$

One of the roots is either +1 or −1. (−1 is a solution.)

Then

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda+1 = -(\lambda+1)(a\lambda^2+b\lambda+c \qquad b=1$$

One way is long division. The second method is to expand RHS and determine the coefficients.

$$\lambda^3-\frac{2}{3}\lambda^2-\frac{2}{3}\lambda-1 = (\lambda+1)(a\lambda^2+b\lambda+c)$$

$$= a\lambda^3+(a+b)\lambda^2+(b+c)\lambda+c$$

$a=1,\ c=1 \quad a+b=-\frac{2}{3} \Rightarrow 1+b=-\frac{2}{3} \Rightarrow b=-\frac{5}{3}$. Then

$$-\lambda^3+\frac{2}{3}\lambda^2+\frac{2}{3}\lambda+1=-(\lambda+1)\left(\lambda^2-\frac{5}{3}\lambda+1\right)$$

$$\lambda_{2,3}=\frac{\frac{5}{3}\mp\sqrt{\frac{25}{9}-4}}{2}=\frac{\frac{5}{3}\mp\sqrt{-\frac{11}{9}}}{2}=\frac{5\mp\sqrt{11}\,i}{6}$$

$$|\lambda_{2,3}|=\frac{25+11}{36}=\underline{1} \quad ✓$$

Absolute value being equal to 1 is satisfied