8.4 Eigenbases

Eigen values $\rightarrow$ Eigenvectors

Now consider a vector in R^2

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now consider $x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in R^2. Are these linearly independent ?

$$a_1 x^{(1)} + a_2 x^{(2)} = 0$$

$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \Rightarrow \begin{matrix} a_1 + a_2 = 0 \\ -a_2 = 0 \end{matrix}$$

Above equations are satisfied if $a_2 = 0, a_1 = 0$. so $x^{(1)}$ & $x^{(2)}$ are linearly independent.

Question : Can we express any $\underset{\sim}{x} \in R^2$ as :

$$\underset{\sim}{x} = a_1 \underset{\sim}{x}^{(1)} + a_2 \underset{\sim}{x}^{(2)}$$

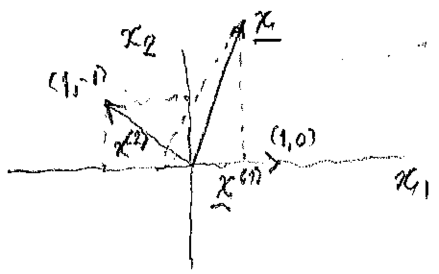
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ -a_2 \end{bmatrix} \Rightarrow \begin{matrix} a_2 = -x_2 \\ a_1 = x_1 - a_2 = x_1 + x_2 \end{matrix}$$

$$= (x_1 + x_2) \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{x}^{(1)}} - x_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underset{\sim}{x}^{(2)}}$$

$\underset{\sim}{x}^{(1)}$ and $\underset{\sim}{x}^{(2)}$ are base vectors ?

Question : Are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ constitute base vectors over R^2 ?

We can find infinite number of base vectors !



We can represent any $\underline{x} \in R^n$ uniquely as a linear combination of the eigenvectors $\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}$.

$$\underline{x} = c_1 \underline{x}^{(1)} + c_2 \underline{x}^{(2)} + \ldots + c_n \underline{x}^{(n)}$$

Now consider $\underline{y} \in R^n \ni \underline{y} = \underline{A}\underline{x}$

$$\underline{y} = \underline{A}\underline{x} = \underline{A}(c_1 \underline{x}^{(1)} + c_2 \underline{x}^{(2)} + \ldots + c_n \underline{x}^{(n)})$$
$$= c_1 \underline{A}\underline{x}^{(1)} + c_2 \underline{A}\underline{x}^{(2)} + \ldots + c_n \underline{A}\underline{x}^{(n)}$$
$$= c_1 \lambda_1 \underline{x}^{(1)} + c_2 \lambda_2 \underline{x}^{(2)} + \ldots + c_n \lambda_n \underline{x}^{(n)}$$

So any vector y is also expressed as the sum of $\underline{x}^{(i)}$; $i = 1, 2, \ldots, n$. Then $\{\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}\}$ form a basis (eigenbasis) set of vectors over R^n.

Example : $\underline{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} -5-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$$

$$\lambda^2 + 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda+1)(\lambda+6) = 0$$

$\lambda_1 = -1$:

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \to \begin{array}{l} -4x_1 + 2x_2 = 0 \\ x_1 = 1 \Rightarrow x_2 = 2 \\ \underline{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

Check: $\underline{A}\underline{x}^{(1)} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \underline{x}$

$\lambda_2 = -6$

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \to \begin{array}{l} x_1 + 2x_2 = 0 \\ x_1 = 2, \Rightarrow x_2 = -1 \\ \underline{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{array}$$

Check

$$\underline{A}\underline{x}^{(2)} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = (-6)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Assume $\underline{x} \in R^2$; $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\underline{x} = c_1 \underline{x}^{(1)} + c_2 \underline{x}^{(2)}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 2c_1 - c_2 \end{bmatrix}$$

Then $c_1 = x_1 - 2c_2$

$$x_2 = 2c_1 - c_2 = 2x_1 - 4c_2 - c_2 = 2x_1 - 5c_2$$

$$\Rightarrow \underline{c_2 = \tfrac{1}{5}(2x_1 - x_2)} \Rightarrow c_1 = x_1 - \tfrac{2}{5}(-x_2 + 2x_1)$$

$$\underline{c_1 = \tfrac{1}{5}x_1 + \tfrac{2}{5}x_2}$$

Check:

$$\underline{x} = c_1 \underline{x}^{(1)} + c_2 \underline{x}^{(2)}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \tfrac{1}{5}(x_1 + 2x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \tfrac{1}{5}(2x_1 - x_2)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \tfrac{1}{5}x_1 + \cancel{\tfrac{2}{5}x_2} + \tfrac{4}{5}x_1 - \cancel{\tfrac{2}{5}x_2} \\ \tfrac{2}{5}x_1 + \tfrac{4}{5}x_2 - \tfrac{2}{5}x_1 + \tfrac{1}{5}x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \checkmark$$

Any vector in R^2 may be expressed as a linear combination of eigenvectors.

Eigenvectors may not be orthogonal or orthonormal.

Basis vectors :

Eigenvectors constitute basis vectors for R^n to express any $\underset{\sim}{x} \in R^n$.

If the basis vectors are orthogonal, then operation on $\underset{\sim}{x}$'s will be much easier.

<u>Theorem</u> : A symmetric matrix has an orthonormal basis of eigenvectors for R^n.

Example :

$$\underset{\sim}{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}, \quad \underset{\sim}{A} \text{ is symmetric.}$$

Eigenvectors : (Found earlier)

$$\underset{\sim}{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underset{\sim}{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\underset{\sim}{x}^{(1)T} \underset{\sim}{x}^{(2)} = [1 \ 2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \ ; \ [\underset{\sim}{x}^{(2)}]^T \underset{\sim}{x}^{2} = [2 \ -1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$|\underset{\sim}{x}^{(1)}| = \sqrt{5}, \quad \underset{\sim}{x}^{(2)} = \sqrt{5}$$

So $\underset{\sim}{x}^{(1)}$ & $\underset{\sim}{x}^{(2)}$ form orthogonal basis.

$$\underset{\sim}{y}^{(1)} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \quad \underset{\sim}{y}^{(2)} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \text{ constitute orthonormal basis}$$

since $|\underset{\sim}{y}^{(1)}| = 1$ & $|\underset{\sim}{y}^{(2)}| = 1$

Definition : An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A if

$$\hat{A} = P^{-1} A P$$

for some nonsingular $n \times n$ matrix P. This transformation to get $\hat{A}$ from A is called similarity transformation

Theorem :

If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigenvalues as A. Consequently if x is an eigenvector of A, then $y = P^{-1} x$ is an eigenvector of $\hat{A}$, corresponding to the same eigenvalue.

$$A x = \lambda x$$

$$P^{-1} A x = P^{-1} \lambda x = \lambda P^{-1} x$$

$$P^{-1} A I x = \underbrace{P^{-1} A P}_{\hat{A}} \cdot \underbrace{P^{-1} x}_{y} = \lambda \underbrace{P^{-1} x}_{y} \Rightarrow \hat{A} y = \lambda y$$

Same eigenvalue λ, but different eigenvector y.

Example :

$$A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = P^{-1} A P = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues :

$$\det(\hat{A} - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) = 0$$

$$\lambda_1 = 3, \; \lambda_2 = 2$$

So the entries are the eigenvalues. Note that these are also the eigenvalues of A.

$\hat{A}$ is a diagonal matrix.

Now determine the eigenvectors for $\bar{A}$

$\lambda_1 = 3 \Rightarrow (\underline{A} - \lambda \underline{I})\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} 6-3 & -3 \\ 4 & -1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $3x_1 - 3x_2 = 0 \Rightarrow x_2 = 1, x_1 = 1$

$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2 \Rightarrow (\underline{A} - \lambda \underline{I})\underline{x} = \begin{bmatrix} 4 & -3 \\ 4 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = 4 \rightarrow x_1 = 3$

$\underline{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Eigenvectors for $\hat{\underline{A}} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\lambda_1 = 3 \quad (\hat{\underline{A}} - \lambda \underline{I})\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0 \Rightarrow x_1 = 1$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \quad (\hat{A} - \lambda \underline{I})\underline{x} = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $x_1 = 0, x_2 = 1$

$\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Check:

$\underline{y} = P^{-1}\underline{x}$

For $\lambda_1 = 3$

$\underline{y} = \underline{P}^{-1}\underline{x} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

$\lambda_2 = 2$

$\underline{y} = \underline{P}^{-1}\underline{x} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

Diagonalization :

<u>Theorem</u>: If an $n \times n$ matrix has a basis of eigenvectors then

$$D = \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X}$$

is diagonal, with the eigenvalues of $\underset{\sim}{A}$ as the entries on the main diagonal. Here $\underset{\sim}{X}$ is the matrix with these eigenvectors as the column vectors.

<u>Example</u>

$$\underset{\sim}{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$$

It is found that $\lambda_1 = 3$, $\lambda_2 = 2$ with eigenvectors

$$\underset{\sim}{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underset{\sim}{x}^{(2)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Then

$$\underset{\sim}{X} = \underset{\sim}{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \underset{\sim}{X}^{-1} = \frac{1}{1}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}, \quad \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

<u>Example</u>

$$\underset{\sim}{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

we found that $\lambda_1 = -1$ & $\lambda_2 = -6$ with

$$\underset{\sim}{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \& \quad \underset{\sim}{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Then

$$\underset{\sim}{X} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow \underset{\sim}{X}^{-1} = \frac{1}{-1-4}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$\underset{\sim}{D} = \underset{\sim}{X}^{-1} \underset{\sim}{A} \underset{\sim}{X} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}\begin{bmatrix} -1 & -12 \\ -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

Example:

$$\underline{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \qquad \lambda_1 = 5, \ \lambda_{2,3} = 3$$

$$\lambda_1 = 5$$

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \overbrace{\underline{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}^{\lambda_{2,3} = 3}$$

$$\underline{X} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \underline{X}^{-1} = \begin{bmatrix} 1/8 & 1/4 & -3/8 \\ -1/4 & 1/2 & 3/4 \\ 1/8 & 1/4 & 5/8 \end{bmatrix}$$

$$\underline{X}^{-1}\underline{A} = \begin{bmatrix} 5/8 & 5/4 & -15/8 \\ 3/4 & -3/2 & -9/4 \\ -3/8 & -3/4 & -15/8 \end{bmatrix}$$

$$\underline{D} = \underline{X}^{-1}\underline{A}\,\underline{X} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Quadratic form :

$$Q = a x_1^2 + b x_1 x_2 + c x_2^2$$

is called quadratic form. Above expression may be written as

$$Q = \underset{\sim}{x}^T \underset{\sim}{A}\, \underset{\sim}{x} = [x_1\ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = [x_1\ x_2]\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2\end{bmatrix} = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2$$

$$Q = \underbrace{[x_1\ x_2]}\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = [x_1\ x_2]\begin{bmatrix} a x_1 + \frac{b}{2}x_2 \\ \frac{b}{2}x_1 + c x_2\end{bmatrix}$$

$$= a x_1^2 + \frac{b}{2}x_1x_2 + \frac{b}{2}x_1x_2 + c x_2^2 = a x_1^2 + b x_1 x_2 + c x_2^2$$

This decomposition results in symmetric $\underset{\sim}{A}$.

Through a proper transposition above quadratic form may be put into the form:

$$Q = a' y_1^2 + c' y_2^2 .$$

Assume a general form : $\underset{\sim}{x} \in R^n$, $\underset{\sim}{A} = n \times n$ matrix

$$Q = \underset{\sim}{x}^T \underset{\sim}{A}\, \underset{\sim}{x}$$

$$\begin{aligned} &= a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\ &+ a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\ &\ \ \vdots \\ &+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2 \end{aligned}$$

$\underset{\sim}{A}$ = coefficient matrix

If $a_{12} = a_{21}$; $a_{13} = a_{31}, \ldots$ $\underset{\sim}{A}$ is symmetric.

Example :

$$Q = [x_1\ x_2]\begin{bmatrix} 3 & 4 \\ 6 & 2\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2 = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= [x_1\ x_2]\begin{bmatrix} 3 & 5 \\ 5 & 2\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}$$ Now $\underset{\sim}{A}$ is symmetric.

For a symmetric matrix $\underline{A}$, orthonormal eigen vectors may defined to form a matrix $\underline{X}$ with colums equal to orthonormal vectors. Then

$$\underline{X}^T = \underline{X}^{-1}$$

$$\underline{D} = \underline{X}^{-1}\underline{A}\,\underline{X} \quad \text{(Diagonal matric consisting of eigenvalues}$$

$$\underline{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Then premultiply with $\underline{X}$

$$\underline{X}\,\underline{D} = \underbrace{\underline{X}\,\underline{X}^{-1}}_{\underline{I}}\underline{A}\,\underline{X} = \underline{I}\,\underline{A}\,\underline{X} = \underline{A}\,\underline{X}$$

Postmultiply by $\underline{X}^{-1}$:

$$\underline{X}\,\underline{D}\,\underline{X}^{-1} = \underline{A}\,\underline{X}\,\underline{X}^{-1} = \underline{A}\,\underline{I} = \underline{A} \Rightarrow \underline{A} = \underline{X}\,\underline{D}\,\underline{X}^{-1}$$

Then

$$Q = \underline{x}^T\underline{A}\,\underline{x} = \underline{x}^T\underline{X}\,\underline{D}\,\underbrace{\underline{X}^{-1}}_{\underline{X}^T}\underline{x} = \underline{x}^T\underline{X}\,\underline{D}\,\underline{X}^T\underline{x}$$

Let $\underline{y} = \underline{X}^T\underline{x} \Rightarrow \underline{y}^T = \underline{x}^T(\underline{X}^T)^T = \underline{x}^T\underline{X}$

Then

$$Q = \underbrace{\underline{x}^T\underline{X}}_{\underline{y}^T}\,\underline{D}\,\underbrace{\underline{X}^T\underline{x}}_{\underline{y}} = \underline{y}^T\underline{D}\,\underline{y}$$

$$= [y_1\ y_2 \cdots y_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow$ Symmetrix

$$Q = 2x_1^2 + 2x_1x_2 + 2x_2^2 = 27$$

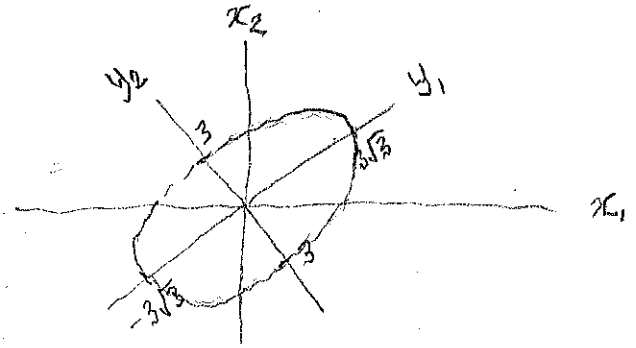
$$\det(\underline{A} - \lambda \underline{I}) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0$$

$$= (2-\lambda-1)(2-\lambda+1)$$

$$= (1-\lambda)(3-\lambda) = 0 \quad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \end{cases}$$

so $Q = y_1^2 + 3y_2^2 = 27$ $\quad Q = \underline{y}^T \underline{D} \underline{y}$, $\underline{D} = \underline{X}^{-1} \underline{A} \underline{X}$, $\underline{y} = \underline{X}^T \underline{x}$

$$\left(\frac{y_1}{\sqrt{27}}\right)^2 + \left(\frac{y_2}{3}\right)^2 = 1 \rightarrow \text{An ellipse}$$



To determine the rotated new transformed values determine the orthonormal basis $= (\underline{A} - \lambda \underline{I})\, \underline{x} = 0$

(i) $\lambda_1 = 1$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = -1,\ x_1 = 1$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow[\text{Divide by } |x|]{\text{orthonormal}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(ii) $\lambda_2 = 3$

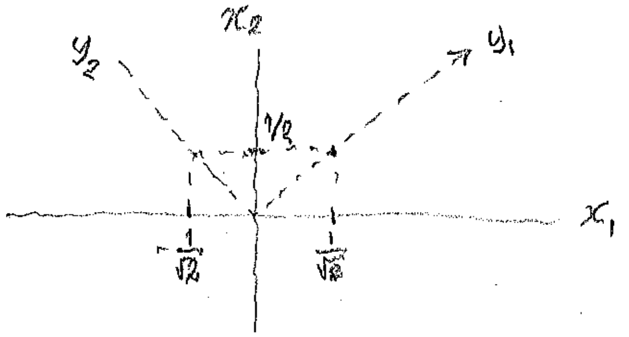
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = 1,\ x_2 = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow[\text{Divide by } |\underline{x}|]{\text{orthonormal}} \begin{bmatrix} 1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\underline{y} = \underline{X}^T \underline{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

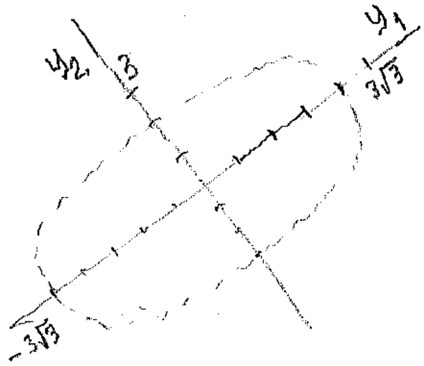
Now let's determine $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ corresponding $\underline{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$



$$\underline{y} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The point $\underline{y}$ corresponding to $\underline{x} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$y = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Gram Schmidt Orthogonalization

Assume $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ constitute Linearly independent set of basis vectors for R^n.

Can we determine $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n$; a set of orthonormal basis for R^n using $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$?

$$\underline{y}_1 = \frac{\underline{x}_1}{\|\underline{x}_1\|}$$

$$\underline{w}_2 = \underline{x}_2 - (\underline{x}_2 \cdot \underline{y}_1)\underline{y}_1 \Rightarrow \underline{y}_2 = \frac{\underline{w}_2}{\|\underline{w}_2\|}$$

$$\underline{w}_3 = \underline{x}_3 - (\underline{x}_3 \cdot \underline{y}_1)\underline{y}_1 - (\underline{x}_3 \cdot \underline{y}_2)\underline{y}_2 \Rightarrow \underline{y}_3 = \frac{\underline{w}_3}{\|\underline{w}_3\|}$$

$$\vdots$$

$$\underline{w}_k = \underline{x}_k - (\underline{x}_k \cdot \underline{y}_1)\underline{y}_1 - (\underline{x}_k \cdot \underline{y}_2)\underline{y}_2 - \cdots - (\underline{x}_k \cdot \underline{y}_{k-1})\underline{y}_{k-1} \Rightarrow \underline{y}_k = \frac{\underline{w}_k}{\|\underline{w}_k\|}$$

Example:

$$\underline{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix},$$

$$\lambda_1 = 3 \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 2 \Rightarrow \underline{x}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\underline{x}_1$ and $\underline{x}_2$ are Linearly independent; they form basis vector set for R^n.

$$\underline{x}_1 \cdot \underline{x}_2 = \underline{x}_1^T \underline{x}_2 = [1 \;\; 1]\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 7 \neq 0 \Rightarrow \underline{x}_1 \,\&\, \underline{x}_2 \text{ are not orthogonal.}$$

Determine an orthonormal set $\{\underline{y}_1, \underline{y}_2\}$ using $\{\underline{x}_1, \underline{x}_2\}$.

$$\underline{y}_1 = \frac{\underline{x}_1}{\|\underline{x}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \text{Not that } \|\underline{y}_1\| = 1.$$

$$\underline{w}_2 = \underline{x}_2 - (\underline{x}_2 \cdot \underline{y}_1)\underline{y}_1 = \underline{x}_2 - (\underline{x}_2^T \underline{y}_1)\underline{y}_1$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} - [3 \;\; 4]\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \frac{7}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$\|\underline{w}_2\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \Rightarrow \underline{y}_2 = \frac{1}{1/\sqrt{2}}\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$\underline{y}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{Note that } \|y_2\| = 1$$

$$\underline{y}_1 \cdot \underline{y}_2 = y_1^T \underline{y}_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 0$$

$\|\underline{y}_1\| = 1$, $\|\underline{y}_2\| = 1$ & $(\underline{y}_1 \cdot \underline{y}_2) = 0 \Rightarrow \{y_1, y_2\}$ orthonormal basis for R^2.

Example:

Leontief Matrix

Price is determined by the price of three components.

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}}_{\text{Leontief matrix}} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \qquad \underline{P} = \underline{A}\,\underline{P}$$

Consider

$$\underline{A}\,\underline{P} = \underline{P} \Rightarrow \underline{A}\,\underline{P} = \lambda \underline{P} \qquad (\lambda \overset{?}{=} 1)$$

$$\det(\underline{A} - \lambda \underline{I}) = 0$$

$$\begin{vmatrix} 0.1-\lambda & 0.5 & 0 \\ 0.8 & -\lambda & 0.4 \\ 0.1 & 0.5 & 0.6-\lambda \end{vmatrix} = 0$$

$$(0.1-\lambda)[-\lambda(0.6-\lambda)-0.2] - 0.5[0.8(0.6-\lambda)-0.04] = 0$$

$$(0.1-\lambda)[\lambda^2-0.6\lambda-0.2] - 0.5[-0.8\lambda+0.48-0.04] = 0$$

$$(-\lambda^3+0.6\lambda^2+0.2\lambda+0.1\lambda^2-0.06\lambda-0.02)+(0.4\lambda-0.22)$$

$$(-\lambda^3+0.7\lambda^2+0.14\lambda-0.02)+(0.4\lambda-0.22) = 0$$

$$-\lambda^3+0.7\lambda^2+0.54\lambda-0.24 = 0$$

$\lambda = 1$ is a solution.

$$\underline{A}\,\underline{P} = \underline{P}$$

$$(\underline{A} - \lambda \underline{I})\underline{P} = 0 \Rightarrow \begin{bmatrix} -0.9 & 0.5 & 0 \\ 0.8 & -1 & 0.4 \\ 0.1 & 0.5 & -0.4 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.9 & 0.5 & 0 \\ 0 & -5 & 0.36 \\ 0 & 5 & -0.36 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.9 & 0.5 & 0 \\ 0 & -5 & 0.36 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

$$-0.9P_1 + 0.5P_2 = 0$$

$$-5P_2 + 0.36P_3 = 0 \Rightarrow P_3 = 1000 \Rightarrow P_2 = \frac{360}{5} = 72$$

$$-0.9P_1 + 0.5P_2 = 0 \Rightarrow P_1 = \frac{(0.5)(72)}{0.9} = 40$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 72 \\ 1000 \end{bmatrix}$$

$$-\lambda^3 + 0.7\lambda^2 + 0.54\lambda - 0.24 \quad \Big| \; \lambda - 1$$

$$\pm \cancel{\lambda^3} \mp \lambda^2 \qquad -\lambda^2 - 0.3\lambda + 0.24$$

$$-0.3\lambda^2 + 0.54\lambda - 0.24$$

$$\pm 0.3\lambda^2 \mp 0.3\lambda$$

$$0.24\lambda - 0.24$$

$$0.24\lambda - 0.24$$

Then

$$-\lambda^3 + 0.7\lambda^2 + 0.54\lambda - 0.24 = (\lambda - 1)(-\lambda^2 - 0.3 + 0.24) = 0$$

$$\lambda_1 = 1, \qquad \lambda_{2,3} = \frac{0.3 \mp \sqrt{0.09 + 0.96}}{-2} = \frac{0.3 \mp \sqrt{1.05}}{-2}$$