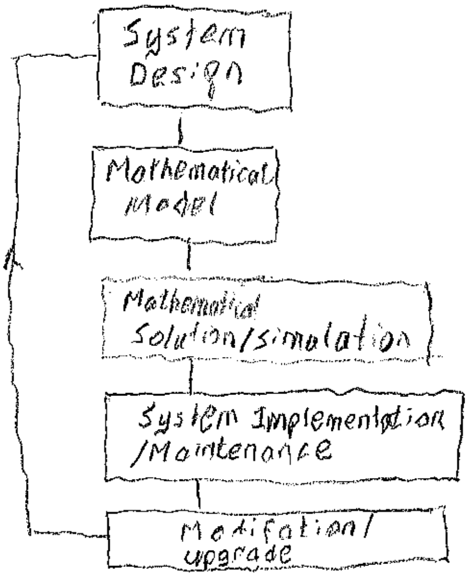
DIFFERENTIAL EQUATIONS

1.1 Basic Concepts: Modeling

To solve engineering problems (starting from design), the components of the system are modeled in terms of variables, functions and equations. This process is called mathematical modeling or just modeling.

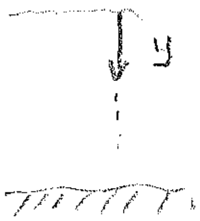
Such a model usually contains derivates, of unknown variables. Such a model is called a differential equation.



$$\frac{dy}{dt} = y'$$

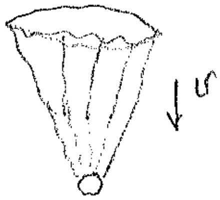
$$\frac{d^2y}{dt^2} = y''$$

$$\frac{dy}{dx}$$



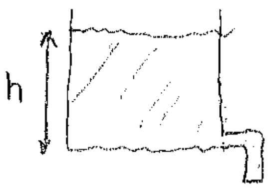
Falling stone:

$y'' = g$ constant $\Big\}$ $\dfrac{d^2y}{dt^2} = g$



$mv' = mg - bv^2$

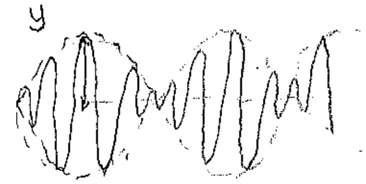
gravity   air resistance



Outflowing water

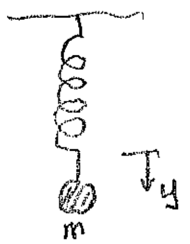
$h' = -k\sqrt{h}$ (k is a constant, depending on the diameter of the pipe.

Vibrating System



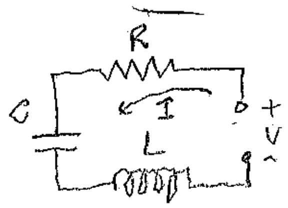
$$y'' + \omega_0^2 y = \cos \omega t \quad \omega_0 \approx \omega$$

Spring



Displacement y:

$$m y'' + k y = 0$$



$$L I'' + R I' + \frac{1}{C} I = V'$$

Examples :

$$\left.\begin{array}{l} y' = \cos x \\ y'' + 9y = e^{-2x} \\ y' y''' - \frac{3}{2}(y')^2 = 0 \end{array}\right\} \quad \frac{dy}{dx} = y'$$

Ordinary differential equations. (ODE)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$$ Partial differential equations (PDE)

What is partial derivative?

$$u = x^2 y$$

$$\frac{\partial u}{\partial x} = 2xy \quad (y \text{ is constant})$$

$$\frac{\partial u}{\partial y} = x^2 \quad (x \text{ is constant})$$

Total derivative:

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

If the system equation contains the first derivative y'; it is called as first order ODEs.

$$F(x,y,y')=0 \rightarrow \text{Implicit form}$$

$$y'=f(x,y) \rightarrow \text{Explicit form}$$

Example :

$$\frac{y'}{x^3}-4y^2=0 \Rightarrow y'=4x^3y^2 \quad (\text{Explicit form})$$

Solution : $y=h(x)$

Assume an ODE is given. Then $y=h(x)$ [or $y=h(t)$] is a solution if it satisfies the given ODE, over the defined domain $x\in(a,b)$.

Example : Assume $xy'=-y \quad \forall x\neq 0$

Show that $y=\frac{c}{x}$ is a solution.

$$y'=-\frac{c}{x^2} \Rightarrow xy'=-y$$

$$\cancel{x}\left(-\frac{c}{x^{\cancel{2}}}\right)=-\frac{c}{x} \Rightarrow -\frac{c}{x}=-\frac{c}{x} \quad x\neq 0$$

c is a constant to be determined by initial condition, or any value of y given

$$y(1)=+1 \Rightarrow 1=c\cdot 1 \Rightarrow c=1$$

Example : $y'=\frac{dy}{dx}=\cos x$

$$dy=\cos x\, dx$$

Integrate both sides :

$$\int dy=\int \cos x\, dx \Rightarrow y=\sin x+c \quad (\text{shifted sine waves})$$

c is to be determined by the initial conditions.

$y(0)=0 \Rightarrow 0=0+c \Rightarrow c=0$

$y(\frac{\pi}{2})=0 \Rightarrow 0=1+c \Rightarrow c=-1$

Example: Growth of bacteria

$$y' = \frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt \Rightarrow \ln y = kt + c_1$$

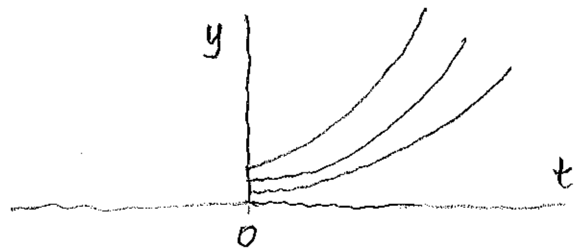
$$y = e^{kt+c_1} = \underbrace{e^{c_1}}_{c} e^{kt} = ce^{kt}$$

Assume

$$y = ce^{at} \Rightarrow y' = ace^{at}$$

$$a\cancel{c}\cancel{e}^{at} = \cancel{c}\cancel{e}^{at}k \Rightarrow a = k$$

Solution is in the form of $y = ce^{kt}$



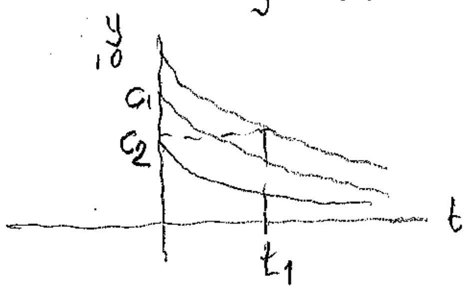
c is to be determined by an $y(t_1)$ value.

$y(0) = 1 \Rightarrow c = 1$

Example: Radioactive decay

$$y' = \frac{dy}{dt} = -ky$$

$$y = ce^{-kt} \quad \text{(Exponential decay)}$$



Let $y(0) = 10 \Rightarrow c = 10$

$y(0) = c$ $\quad$ (t_1 = Half life)

$y(t_1) = \frac{c}{2}$

$$y = ce^{-kt_1} = \cancel{c}e^{-kt_1} = \frac{\cancel{c}}{2}$$

$$e^{-kt_1} = \frac{1}{2} \Rightarrow e^{kt_1} = 2$$

$$t_1 = \frac{1}{k}\ln 2 = \frac{0.69}{k}$$

General Solution: Solution containing $c_1, c_2, \dots$

Particular Solution: Solution with determined and known c_i's, using the conditions given (Initial conditions, boundary conditions)

1.2. Numeric Method by Euler (Computer Simulations)

$$y' = f(x, y) \qquad y(x_0) = y_0$$

$$y'(x_0) = f(x_0, y_0)$$

$$\frac{y(x_0+h) - y(x_0)}{h} = f(x_0, y_0)$$

$$y(x_0+h) = y(x_0) + h f(x_0, y_0) \rightarrow y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

Or in general,

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

h : step size : If $h \searrow$ Then computer time $\nearrow$

$h \nearrow$ " " $\searrow$

Example : $y' = y + x \qquad y(0) = 0$, $x_0 = 0$

$f(x, y) = y + x \qquad$ Exact solution : $y = e^x - x - 1$

$h = 0.2$

n	x_n	y_n	y_{n+1}	$y_{n\text{-}actual}$	Error
0	0	0	0	0	0
1	0.2	0	0.04	0.021	0.021 ←
2	0.4	0.04	0.128	0.092	0.052
3	0.6	0.128	0.274	0.222	0.094
4	0.8	0.274	0.488	0.426	0.152
5	1.0	0.488	0.786	0.718	0.230 ←

$h = 0.1$

n	x_n	y_n	y_{n+1}	$y_{n\text{-}actual}$	Error
0	0.0	0	0	0	0
1	0.1	0.0	0.01	0.005	0.005
2	0.2	0.01	0.031	0.021	0.011 ←
⋮					
10	1.0	0.594	0.753	0.718	0.125 ←

1.3 Separable ODEs.

Assume ODE can be put into the following form:

$$g(y)\,y' = f(x)$$

$$g(y)\frac{dy}{dx} = f(x) \Rightarrow \int g(y)\,dy = f(x)\,dx$$

$$\int g(y)\,dy = \int f(x)\,dx$$

<u>Example:</u> $y' = 1 + y^2$

$$\frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = dx$$

$$\int \frac{dy}{1+y^2} = \int dx \Rightarrow \tan^{-1} y = x + c \Rightarrow y = \tan(x+c)$$

<u>Example:</u> $y' = (x+1)e^{-x}y^2$

$$\frac{dy}{dx} = (x+1)e^{-x}y^2 \Rightarrow \frac{dy}{y^2} = (x+1)e^{-x}dx$$

$$\int \frac{dy}{y^2} = \int (xe^{-x} + e^{-x})\,dx = \int xe^{-x}dx + \int e^{-x}dx$$

$$\int xe^{-x}dx = ? \qquad u = x,\ dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int u\,dv = uv - \int v\,du \Rightarrow \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x}$$

$$\int \frac{dy}{y^2} = \int xe^{-x}dx + \int e^{-x}dx$$

$$-\frac{1}{y} = -xe^{-x} - e^{-x} - e^{-x} + c \Rightarrow \frac{1}{y} = (x+2)e^{-x} - c$$

$$y = \frac{1}{(x+2)e^{-x} - c}$$

Example: $y' = -2xy \quad y(0) = 2$

$$\frac{dy}{dx} = -2xy \Rightarrow \frac{dy}{y} = -2x\,dx$$

$$\ln y = -\frac{2}{2}x^2 = -x^2 + c' \Rightarrow y = e^{(-x^2+c')} = e^{-x^2}\cdot\underbrace{e^{c'}}_{c} = ce^{-x^2}$$

$$y = ce^{-x^2} \;;\; y(0) = 2 \Rightarrow 2 = c\cdot 1 \Rightarrow c = 2$$

$$y = 2e^{-x^2}$$

Example: Radioactive decay:

$$y' = -ky \Rightarrow \frac{dy}{dt} = -ky$$

$$\frac{dy}{y} = -k\,dt \Rightarrow \ln y = -kt + c' \Rightarrow y = e^{-kt+c'} = e^{-kt}c$$

$$y(0) = y_0 = c \Rightarrow y(t) = y_0 e^{-kt}$$

$k = 0.001$ 1/years determine the half life:

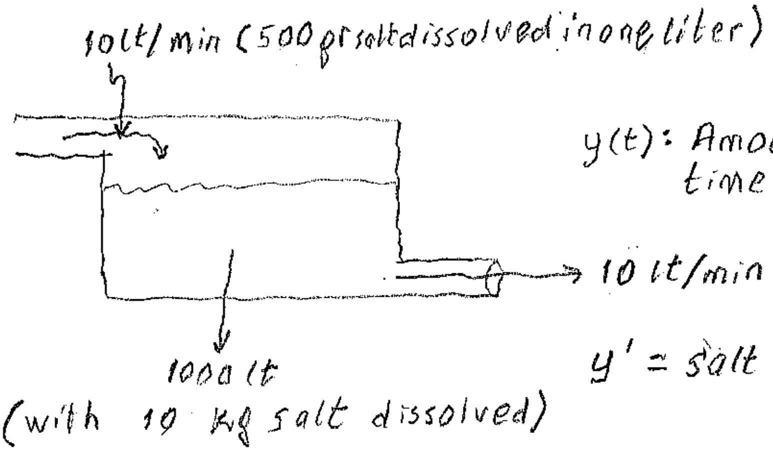
$$y(t_h) = \frac{y_0}{2} = y_0 e^{-0.001t} \Rightarrow \frac{1}{2} = e^{-0.001t_h} = \frac{1}{e^{0.001t_h}}$$

$$\Rightarrow e^{0.001t_H} = 2 \Rightarrow 0.001t_h = \ln 2 = 0.693$$

$$t_h = 693 \text{ years}$$

Example

5 lb → 500gr
gal → lt



$y(t)$: Amount of salt in the tank at time t.

$y' =$ salt inflow rate − Salt outflow rate

Incoming liquid has $(10\, lt/min)(0.5\, kg/lt) = 5\, kg/min$ (constant)

Outflow is 10 lt/min) that contains $\frac{10\, kg}{1000\, lt} = 0.01\, kg/lt$ salt.

Hence each minute $\frac{10 lt}{1000 lt} = 0.01$ portion of the tank is outgoing, which then implies "$0.01\, y$" salt is leaving the tank, since the tank is stirred and salt is uniformly disrubuted. Then

$$y' = 5 - 0.01y = -0.01(y-500) \quad kg/min.$$

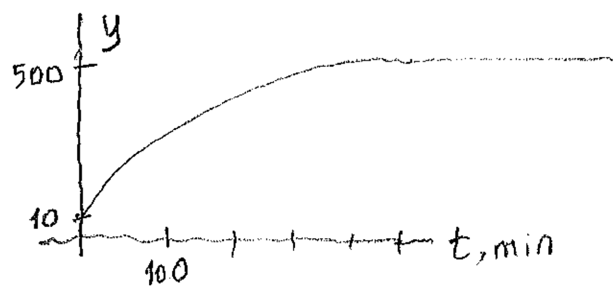
$$\frac{dy}{dt} = -0.01(y-500) \Rightarrow \int \frac{dy}{y-500} = \int -0.01t$$

$$\ln(y-500) = -0.01t + c' \quad y - 500 = e^{(-0.01t + c')}$$

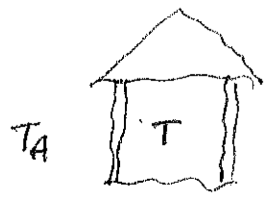
$$y - 500 = e^{-0.01t} \cdot e^{c_1} = ce^{-0.01t} \Rightarrow y = 500 + ce^{-0.01t}$$

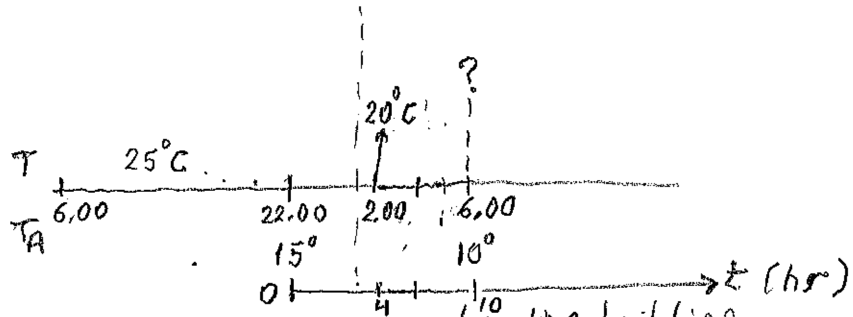
Initially $y(0) = 10$ kg

$$10 = 500 + c \Rightarrow c = -490 \Rightarrow y = 500 - 490e^{-0.01t}$$



Example :





T : temperature inside the building

T_A : " outside " " (ambient temp.)

Newtons law : $\frac{dT}{dt} = k(T - T_A)$ $\{ T_A = 12.5°C$

Since T_A is also time varying, it is not a separable ODE. Make a reasonable assumption that $T_A \cong \frac{15+10}{2} = 12,5°C$ as constant. Also let 22.00 PM as $t = 0$.

$T(0) = 25°C$

$t = 0 \rightarrow T = 25$
$t = 4 \rightarrow T = 20$
$t = 10 \rightarrow T = ?$

$\frac{dT}{T - T_A} = k\,dt \Rightarrow \ln(T - T_A) = kt + c'$

$T - T_A = e^{+(kt + c')} = e^{+kt} \cdot \underbrace{e^{-c'}}_{c} = c\,e^{+kt}$

$T = T_A + c\,e^{+kt}$

$T(0) = 25 \Rightarrow 25 = 12.5 + c \cdot e^0 \Rightarrow c = 12.5$

$T = 12.5 + 12.5 e^{+kt} = 12.5(1 + e^{kt})$

$T(4) = 20 = 12.5(1 + e^{4k}) \Rightarrow \frac{20}{12.5} - 1 = e^{4t}$

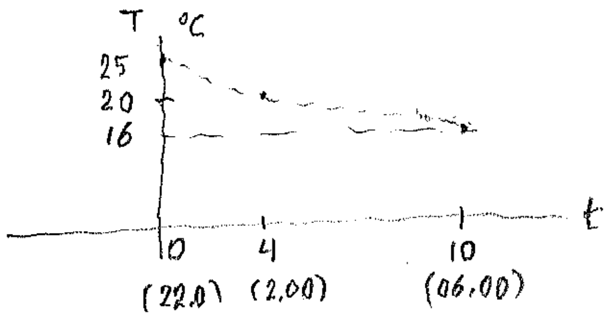
$0.6 = e^{4k} \Rightarrow \ln(0.6) = 4k \Rightarrow$

$-0.511 = 4k \Rightarrow k = -0.128$

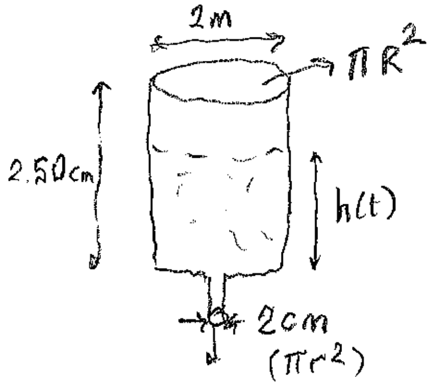
Then $T = 12.5 + 12.5 e^{-0.128t}$

$t = 10$

$T \cong 16°C$



Example: Outflow of water Through a hole. Initially it is full.
$R = 1\,m,\ r = 1\,cm$



$v(t) = 0.6\sqrt{2gh}$

$g = 980\ cm/sec^2$

Determine the time at which tank is empty.

The volume ΔV of the outflow during a short time Δt:

$\Delta V = A v \Delta t \qquad A$: Area of hole

$\Delta V = -B\,\Delta h \Rightarrow \quad B$: Area of the tank $= \pi R^2$

$-B\Delta h = A v \Delta t$

$-(\cancel{\pi} R^2)\Delta h = \cancel{\pi} r^2\, 0.6\sqrt{2gh}\,\Delta t$

$\frac{\Delta h}{\Delta t} = -(0.6)\left(\frac{r}{R}\right)^2 \sqrt{2g}\cdot\sqrt{h}$

Take the limit as $\Delta t \to 0$

$h' = -(0.6)\left(\frac{r}{R}\right)^2\sqrt{2g}\sqrt{h} = -(26.56)\left(\frac{1}{100}\right)^2\sqrt{h}$

$\frac{h'}{\sqrt{h}} = -26.56\times10^{-4} \Rightarrow \frac{dh}{\sqrt{h}} = -2.656\times10^{-3}dt$

$2h^{1/2} = -2.656t\times10^{-3}t + c$

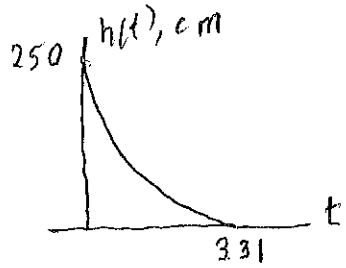
$h(t) = (c - 1.328\times10^{-3}t)$

$h(0) = 250\ cm \Rightarrow 250 = c^2 \Rightarrow c = 15.811$

$h(t) = (15.811 - 1.328\times10^{-3}t)^2$

$h(t_f) = 0 \Rightarrow 15.811 - 1.328\times10^{-3}t_f = 0$

$t_f = 11905\ sec = 3.31\ hrs$



Reduction to Separable Form

$$y' = f\left(\frac{y}{x}\right)$$

Can we put such a differential equation into a separable form? $\frac{y}{x} = u$

Let $y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u \Rightarrow y' = u'x + u$

$$y' = f\left(\frac{y}{x}\right) = f(u)$$

$$u'x + u = f(u) \Rightarrow \frac{du}{dx} x = f(u) - u$$

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

Example: $2xyy' = y^2 - x^2 \Rightarrow y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$

$$\left.\begin{array}{l} u = \frac{y}{x} \Rightarrow y' = \frac{u}{2} - \frac{1}{2u} \\ \\ y = ux \Rightarrow y' = u'x + u \end{array}\right\} \begin{array}{l} u'x + u = \frac{u}{2} - \frac{1}{2u} \\ u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u} \end{array}$$

$$\frac{du}{dx} x = \frac{-u^2 - 1}{2u} \Rightarrow -\frac{2u}{u^2 + 1} du = -\frac{1}{x} dx$$

$$v = u^2 + 1 \rightarrow dv = 2u\,du$$

$$\frac{dv}{v} = \frac{-dx}{x} \Rightarrow \ln v = -\ln x + c$$

$$\ln(1 + u^2) = -\ln x + c = \ln\left(\frac{1}{x}\right) + c' = \ln\left(\frac{c}{x}\right)$$

$$1 + u^2 = \frac{c}{x} \Rightarrow 1 + \left(\frac{y}{x}\right)^2 = \frac{c}{x} \Rightarrow \frac{x^2 + y^2}{x^2} = \frac{c}{x}$$

$$\Rightarrow x^2 + y^2 = cx \Rightarrow x^2 - cx + \left(\frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$$

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$$