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A Graph-theoretic perspective on centrality[☆]

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Abstract

The concept of centrality is often invoked in social network analysis, and diverse indices have been proposed to measure it. This paper develops a unified framework for the measurement of centrality. All measures of centrality assess a node's involvement in the walk structure of a network. Measures vary along four key dimensions: type of nodal involvement assessed, type of walk considered, property of walk assessed, and choice of summary measure. If we cross-classify measures by type of nodal involvement (radial versus medial) and property of walk assessed (volume versus length), we obtain a four-fold polychotomization with one cell empty which mirrors Freeman's 1979 categorization. At a more substantive level, measures of centrality summarize a node's involvement in or contribution to the cohesiveness of the network. Radial measures in particular are reductions of pair-wise proximities/cohesion to attributes of nodes or actors. The usefulness and interpretability of radial measures depend on the fit of the cohesion matrix to the one-dimensional model. In network terms, a network that is fit by a one-dimensional model has a core-periphery structure in which all nodes revolve more or less closely around a single core. This in turn implies that the network does not contain distinct cohesive subgroups. Thus, centrality is shown to be intimately connected with the cohesive subgroup structure of a network.

1. Introduction

Centrality is a fundamental concept in network analysis. Bavelas (1948, 1950) and Leavitt (1951) used centrality to explain differential performance of communication networks and network members on a host of variables including time to problem solution, number of errors, perception of leadership, efficiency, and job satisfaction. Their work led to a great deal of

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experimental, empirical, and theoretical research on the implications of network structure for substantive outcomes, particularly in the context of organizations. Centrality has been used to investigate influence in interorganizational networks (Laumann and Pappi, 1976; Marsden and Laumann, 1977; Galaskiewicz, 1979), power (Burt, 1982; Knoke and Burt, 1983), advantage in exchange networks (Cook et al., 1983; Marsden, 1982), competence in formal organizations (Blau, 1963), employment opportunities (Granovetter, 1974), adoption of innovation (Coleman et al., 1966), corporate interlocks (Mariolis, 1975; Mintz and Schwartz, 1985; Mizruchi, 1982), status in monkey grooming networks (Sade, 1972, 1989), power in organizations (Brass, 1984) and differential growth rates among medieval cities (Pitts, 1979). In addition, many other studies use well-known measures of centrality but do not identify them as such. For example, researchers working with ego-networks use the term “network size” (Campbell et al., 1986; Deng and Bonacich, 1991) to refer to a variable that in another context we would recognize as degree centrality.

While many measures of centrality have been proposed, the category itself is not well defined beyond general descriptors such as node prominence or structural importance. In addition, people propose all kinds of interpretations of centrality measures, such as (potential for) autonomy, control, risk, exposure, influence, belongingness, brokerage, independence, power and so on. The one thing that all agree on is that centrality is a node-level construct. But what specifically defines the category? What do all centrality measures have in common? Are there any structural properties of nodes that are not measures of centrality?

Sabidussi (1966) tried to provide a mathematical answer to these questions. He suggested a set of criteria that measures must meet in order to qualify as centrality measures. For example, he felt that adding a tie to a node should always increase the centrality of the node, and that adding a tie anywhere in the network should never decrease the centrality of any node. These requirements are attractive: it is easy to see the value of separating measures that are “well-behaved” from measures that behave less intuitively. However, there are problems with Sabidussi’s approach. For one thing, it turns out that his criteria eliminate most known measures of centrality, including betweenness centrality. This is clearly unsatisfactory. Furthermore, while his criteria provide some desirable, prescriptive characteristics for a centrality measure, they do not actually attempt to explain what centrality is.

Freeman (1979) provided another approach to answering the ‘what is centrality’ question. He reviewed a number of published measures and reduced them to three basic concepts for which he provided canonical formulations. These were degree, closeness and betweenness. He noted that all three attain their maximum values for the center of a star-shaped network. It can be argued that this property serves as a defining characteristic of proper centrality measures.

Borgatti (2005) has recently proposed a dynamic model-based view of centrality that focuses on the outcomes for nodes in a network where something is flowing from node to node across the edges. He argues that the fundamental questions one wants to ask about individual nodes in the dynamic flow context are (a) how often does traffic flow through a node and (b) how long do things take to get to a node. Once these questions are set, it becomes easier to construct graph-theoretic measures based on the structure of the network that predict the answers to these questions. Hence, in this approach, measures of centrality are cast as predictive models of specific properties of network flows.

In this paper, we present an alternative perspective that eschews the dynamic element and is fundamentally structural in character. It is a graph-theoretic review of centrality measures that classifies measures according to the features of their calculation. Whereas the model-based view is centered on the outcomes of centrality, the graph-theoretic view is centered on the way centrality measures are calculated. In short, the present perspective is a means-based classification rather than the ends-based classification presented by Borgatti (2005).

2. Terminology

For simplicity (and in accordance with centrality convention), we will assume that all networks on which we might compute centrality measures consist of undirected graphs $G(V, E)$, in which V is a set of nodes (also called vertices, points or actors) and E is a set of edges (also called ties or lines) that connect them. Many centrality measures can be discussed in terms of directed graphs as well, but this topic is not treated here. It will be helpful to represent a graph in terms of its adjacency matrix A, in which $a_{ij} = 1$ if (i, j) is in E.

Nodes that are not adjacent may nevertheless be reachable from one to the other. A walk from node u to node v is a sequence of adjacent nodes that begins with u and ends with v. A trail is a walk in which no edge (i.e., pair of adjacent nodes) is repeated. A path is a trail in which no node is visited more than once.

The length of a walk is defined as the number of edges it contains, and the shortest path between two nodes is known as a geodesic. The length of a geodesic path between two nodes is known as the geodesic or graph-theoretic distance between them. We can represent the graph theoretic distances between all pairs of nodes as a matrix D in which d_{ij} gives the length of the shortest path from node i to node j.

3. Comparison of methods

To explain the graph-theoretic perspective, we begin by considering a sample of centrality measures and examining how they are computed. In a process similar to the anthropological technique of componential analysis, we extract dimensions along which measures vary. These are then used to develop a three-way typology of measures. We organize the discussion around the three best-known measures of centrality: degree, closeness and betweenness (Freeman, 1979).

3.1. Degree-like measures

We begin by considering one of the simplest and best-known measures of centrality: degree centrality. As defined by Freeman (1979), degree centrality is a count of the number of edges incident upon a given node. As shown in Eq. (1), it can be computed as the marginals of the adjacency matrix A:

$$c_i^{\mathrm{DEG}} = \sum_j a_{ij} \tag{1}$$

We can express this in matrix notation as $C^{\mathrm{DEG}} = A\mathbf{1}$, where $\mathbf{1}$ is a column vector of ones.

It is useful to recognize that every edge is a walk of length 1. Consequently, we can think of degree centrality as counting the number of paths of length 1 that emanate from a node.[2] Degree centrality is therefore a special case of the measure proposed by Sade (1989) called *k*-

[2] As noted earlier, we assume an undirected graph. Hence, we can equally well describe degree in terms of the number of paths of length 1 that terminate at a node.

path centrality[3] which counts all paths of length k or less that emanate from a node. When $k = 1$ (its minimum value), the measure is identical to degree centrality. When $k = n - 1$ (its maximum value), the measure counts the total number of paths of any length that originate at a given node.

In order to establish the commonality of structure across measures of centrality, it is useful to note that k-path centrality may be computed as the marginals of a matrix W in which w_{ij} is the number of paths of length k or less from node i to node j. That is, $C^{\mathrm{K-path}} = W\mathbf{1}$.

Other variations on this theme may be obtained by choosing different restrictions on the kinds of paths counted. For example, if we are only interested in shortest paths, we can define *geodesic k-path centrality* as the number of *geodesic* paths up to length k emanating from a given node. We can think of this as measuring the amount of direct involvement that a node has in the geodesic structure of the network.

Another variation is to count only edge-disjoint paths. Edge-disjoint paths are paths which share no edges. Counting the number of edge-disjoint paths up to length k that originate or terminate at a given node yields a centrality measure we shall call *edge-disjoint k-path centrality*. Disjoint k-path centrality measures can be thought of as inverse measures of vulnerability. This interpretation is based on a theorem by Ford and Fulkerson (1956) which states that the number of edge-disjoint paths linking two nodes is equal to the minimum number of edges that must be deleted in order to disconnect the two nodes.[4] In a network in which ties are subject to destruction (as in roads in a war zone), a disjoint k-path centrality measure assesses how difficult it would be to isolate a given node.

A variant of disjoint k-path centrality counts the number of vertex-disjoint paths up to length k rather than edge-disjoint paths. Vertex-disjoint paths are those which share no vertices (except the two end nodes). The set of such paths in a graph is a subset of the set of edge-disjoint paths. Menger (1927) showed that the number of vertex-disjoint paths linking two nodes is equal to the number of nodes that must be removed from a graph in order to isolate the two nodes from each other. The measure of social proximity developed by Alba and Kadushin (1976), and used as a basis for detecting *social circles*, is a (normalized) count of all vertex-disjoint paths of length 2 or less connecting any two nodes. We call a measure counting the number of vertex-independent paths that originate or terminate at a given node a *vertex-disjoint k-path centrality* measure. The GPI power measure of Markovsky et al. (1988) is a vertex-disjoint k-path centrality measure, which subtracts the number of even-length vertex-disjoint paths emanating from a node from the number of odd-length vertex-disjoint paths emanating from the same node. All *reachability* measures (Higley et al., 1991), which count the number of nodes a given node can reach in a given number of links, are vertex-disjoint k-path centrality measures.

Thus far, we have only considered variations of degree centrality which count true graph-theoretic paths. However, a number of measures count all walks, including those that visit the same nodes repeatedly. Katz's (1953) measure of centrality is a weighted count of the number of walks originating (or terminating) at a given node. The walks are weighted inversely by their length so that long, highly indirect walks count for little, while short, direct walks count for a great deal. The extent to which the weights attenuate with length is controlled by an arbitrary

[3] Sade actually used the term *n-path centrality*, but since n is usually reserved for the number of nodes in a network, we have used k instead.

[4] The number of edge-disjoint paths between two nodes is also equal to the maximum flow between them (Ford and Fulkerson, 1962).

parameter supplied by the researcher. Katz's measure is defined as follows:

$$w_{ij} = ba_{ij} + b^2(a^2)_{ij} + \cdots + b^k(a^k)_{ij} + \cdots = \sum_{k=1}^{\text{infinity}} b^k(a^k)_{ij}$$
$$c_i = \sum_j w_{ij} \tag{2}$$

In matrix notation, $C^{\text{KATZ}} = W\mathbf{1}$. Eq. (2) is based on the fact that the number of walks of length k between all pairs of nodes is given by the kth power of the adjacency matrix. The series is guaranteed to converge only if b is chosen to be smaller than the reciprocal of the largest eigenvalue of A. Hubbell (1965) proposed a measure very similar to Katz's, but which allows for the possibility of taking a weighted row sum of W. The weights are potentially but not necessarily derived from the network itself. If the weighting vector e is chosen to be all ones, Hubbell's measure equals Katz's minus 1. If e is chosen to be the degree of each node, as Hoede (1978) suggested, the result is that Katz's and Hubbell's measures are identical. Friedkin (1991) has developed a measure called *total effects centrality* which is equal to Katz's divided by the constant $1 - b$.

Bonacich (1987) writes a variant of Katz's measure in slightly more general terms as follows:

$$w_{ij} = \delta(a_{ij} + b(a^2)_{ij} + b^2(a^3)_{ij} + \cdots) = \delta \sum_{k=1}^{\text{infinity}} b^k(a^{k+1})_{ij}$$
$$c_i = \sum_j w_{ij} \tag{3}$$
$$C = W1$$

When b is positive, Bonacich's and Katz's measures are perfectly correlated. When b is negative, the two measures are perfectly negatively correlated. A key contribution of Bonacich's was to realize that b could be negative and that this would have a substantive interpretation in exchange networks (Cook et al., 1983; Markovsky et al., 1988). Indeed, Bonacich's measure predicts power use in experimental exchange networks very nicely. This is interesting because a negative value for b means that Eq. (3) effectively subtracts the number of even-length walks from the number of odd-length walks. This is exactly the same as the other well-known measure of power that emerges from the experimental exchange network literature, the GPI index of Markovsky et al. As Markovsky et al., point out, having many alters one link away from a node enhances that node's bargaining power, but having many alters two links away enhances the power of the node's first-order alters, and so on. More generally, a basic principle in exchange networks is that a node is powerful to the extent that it is connected to weak alters. In turn, a node is weak if it is connected to powerful alters. Interestingly, these descriptions resemble the hub and bridge distinctions of Mintz and Schwartz (1981a,b) and Mizruchi et al. (1986). The principal difference between GPI and the Bonacich power measures is that the former counts only vertex-disjoint paths while the latter counts all walks (weighted inversely by length).

Another way of interpreting the walk-based measures is in terms of an intuitive notion[5] that a person's centrality should be a function of the centrality of the people he or she is associated with. In other words, rather than measure the extent to which a given actor "knows everybody",

[5] Probably originating with Alexander (1963), but clearly evident in Bonacich (1972) as well.

we should measure the extent to which the actor "knows everybody who is anybody". Hubbell's measure can be written as follows:

$$c^{\mathrm{HUB}} = Xc^{\mathrm{HUB}} + e \tag{4}$$

where X is a matrix derived from A, and e is the weighting vector of possibly exogenous contributions to status. Katz's and Hoede's measures are a special case of Hubbell's in which e is equal to the row sums of X, and $X = bA$. Thus, both Katz and Hubbell can be seen as "implicit" centrality measures in which the centrality of a node is given by the weighted row sums of an adjusted adjacency matrix, where the weights are the centralities of the columns.

Bonacich (1972) noted the similarity of Eq. (4) to the definition of an eigenvector (Eq. (5)) and recommended that the principal eigenvector (associated with the largest eigenvalue) be used as a centrality measure. He has shown (Bonacich, 1991) that the eigenvector of A is the limit of Katz's measure as b approaches $1/\lambda$ from below. Thus, the eigenvector can be regarded as an elegant summary of Katz's, Hoede's and Hubbell's measures.

$$v = \lambda^{-1} A v \tag{5}$$

Having defined the eigenvector v of adjacency matrix A, we can calculate the W matrix in Eq. (3) more simply, as follows:

$$w_{ij} = a_{ij} v_j \tag{6}$$

Coleman's (1973) Power and Burt's (1982) prestige are applications of the eigenvector measure to specific types of data. In Coleman's case, the matrix A' is restricted to the relation "depends on", while in Burt's case A' is a non-symmetric relation such as "likes".[6]

It is apparent that the variations among the degree-based measures are due entirely to the kinds of restrictions placed on the kinds of walks counted. This defines one typological dimension that we can use to classify measures. We refer to this dimension as *Walk Type*.

3.2. Closeness-like measures

It is also apparent that all of the measures considered so far count the number or volume of walks (of some kind) joining each node to all others. We shall refer to these as *volume measures*. Another set of centrality measures assesses the *lengths* of the walks that a node is involved in. We call these *length measures*. The distinction between volume measures and length measures forms another classificatory dimension, which we call *Walk Property*. It refers to what property of paths (their number or their length) is being measured.

The best-known distance measure is Freeman's (1979) *closeness centrality*, which is defined as the total geodesic distance from a given node to all other nodes. As shown in Eq. (7), it is computed as the marginals of a geodesic distance matrix D:

$$c_i^{\mathrm{CLO}} = \sum_j d_{ij} \tag{7}$$

In matrix notation $C^{\mathrm{CLO}} = D\mathbf{1}$. This is clearly parallel to the degree-based measures discussed in the previous section, with D playing the role of W. Since the number of nodes is fixed in a network,

[6] The descriptions are written in terms of A' instead of A because Coleman and Burt take column sums rather than row sums as we do here.

the measure is equivalent to the mean distance of a node to other nodes. Closeness centrality is an inverse measure of centrality since larger values indicate less centrality. In this sense, it technically measures farness rather than closeness[7].

Direct measures of closeness (rather than farness) can be obtained by transforming the distance matrix into a "nearness" matrix prior to computing row marginals. For example, Høivik and Gleditsch (1975) recommend a linear transformation, as do Valente and Foreman (1998). The latter approach is to take either the row sums (which they call radiality) or the column sums (integration) of the geodesic distance matrix subtracted from a constant.[8] In contrast, Burt (1991) recommends[9] the following exponential transformation:

$$s_{ij} = \alpha^{d_{ij}} \tag{8}$$

Other variants of closeness can be obtained by varying the way the initial distance matrix is defined. In Freeman's measure, closeness is based on geodesic distances. Each d_{ij} entry in the geodesic distance matrix can be viewed as the minimum of the vector of lengths of all paths from i to j. However, if we do not believe that a given substantive phenomenon, such as diffusion of information, always makes use of the shortest paths, it makes sense to take into account all paths from i to j, perhaps by taking the median or mean length of all paths. The latter option is in fact the approach taken by Friedkin (1991) in developing his *immediate effects centrality*, which is defined as the reciprocal of the average distance from a given node to all others, where the distance between two nodes is defined (apart from scaling constants) as the average length of all paths between them. The problem with this, as we have discussed for other measures, is that many of the paths we shall be averaging together will not be totally distinct from each other. The question is, should we give full weight to all paths, or should we try to take into account the fact that some paths are largely redundant?

If we think of each path from i to j as a vector, we can see that what we are looking for is the length of a linear combination of the vectors in which some vectors are weighted more heavily than others according to their distinctiveness. Thus, we seek a set of weights or coefficients that are optimal with respect to some well-specified criterion. The question is, what criterion? In a way, the linear combination we want is the opposite of a factor or principal component. As Nunnally (1967) observes, the variance of a linear combination is high if the vectors are highly (positively) correlated, and low if they are not. Hence, we are looking for a linear combination that has as little variance as possible, given some constraint on the weights. If we denote the kth path between two nodes as p_{k}, the variance of the linear combination $w_1 p_1 + w_2 p_2 + \cdots w_k p_k = \Sigma_k w_k p_k$ is given by

$$Var\left(\sum_k w_k p_k\right) = \sum_k \sum_l w_k w_l \sigma_{kl} \tag{9}$$

where σ_{kl} refers to the covariance between the kth and lth paths. Thus, we seek a set of weights w that minimize Eq. (9), subject to the constraint that $\Sigma w_k = 1$. After differentiating and rearranging

[7] A normalized version of closeness, in which the reciprocals of C^{CLO} are multiplied by the number of nodes minus 1, solves this terminological problem.

[8] It is sometimes claimed that this linear transformation enables us to measure closeness in disconnected graphs (i.e., those containing undefined distances) but this is not the case. In fact, if the constant is taken to be n, this approach gets the same results (linearly rescaled) as simply replacing undefined distances with n, which is clearly unsatisfactory.

[9] In somewhat different context.

terms, we find that the optimal weights are the row marginals of the inverse of the covariance matrix, divided by the sum of all entries. Using these weights, we can theoretically construct a combined path from i to j whose length can be used as the distance between i and j. Computing this distance for all pairs of nodes, a new measure of closeness centrality can be constructed by computing the row marginals of the distance matrix, or, as above, from the reciprocals of the distance matrix.

The difficulty in all this, of course, is that we have not said how paths are to be represented as vectors. One possibility is a 0/1 indicator matrix X in which paths are columns, rows are edges, and values x_{ij} of the matrix indicate whether or not the ith edge occurs in the jth path. Note that the length of a path is given by the column sums minus 1. We can then compute covariance in the usual way, solve for the optimal weights, and construct a minimum variance linear combination. The length of the combined path is then given by the sum of its values.

A different approach is taken by Stephenson and Zelen (1989), who propose that we simply declare the covariance of two paths to be the number of edges they have in common[10]. If we pretend that Eq. (9) still holds, we can then solve for the linear combination of paths with minimum "variance". Further, the variance of the best linear combination is then interpreted as its length, and therefore gives us the distance between i and j. The distances are converted to "nearnesses" by taking reciprocals, and a closeness measure is constructed by taking the harmonic mean of each row of the nearness matrix. Stephenson and Zelen invoke information theory to interpret the nearness matrix as "information", and so they name their measure "information centrality".

A variation on closeness is what we shall call *centroid centrality*. The idea is that one first identifies one or more nodes as the network *centroid*. Then, to calculate the centrality of any node, one measures the distance from that node to the centroid. One obvious choice for the centroid is the graph-theoretic center (Harary, 1969), which is the node (or pair of nodes if not unique) that has the least eccentricity. A node's eccentricity is the length of its longest geodesic path to another node. We can then measure closeness as a node's geodesic distance from the center. Of course, any criteria could be used to identify the central node(s), including other centrality measures. Another approach is to embed the graph into a multidimensional metric space (Freeman, 1983), find the location least distant from all nodes, and use the distance from that point to all others as centrality. This approach was used by Laumann and Pappi (1973). Here, the term "distance" refers to Euclidean distance, or any other distance metric used to define the vector space.

3.3. Betweenness-like measures

All of the measures considered so far—including both the volume and the length measures—assess walks that emanate from or terminate with a given node. We shall refer to these as *radial* measures. Another class of centrality measures exists which are based on the number

[10] Interestingly, when all paths are the same length, the covariance between paths (represented by the edge-path incidence matrix described previously) really is related to the number of edges they have in common. Recall that covariance between paths k and l is defined as

$$\frac{1}{n}\sum_{i}^{n} x_{ik}x_{il} - \left(\frac{1}{n}\sum_{i}^{n} x_{ik}\right)\left(\frac{1}{n}\sum_{i}^{n} x_{il}\right)$$

When all paths have the same length, the term to the right of the subtraction becomes a constant, so the covariance is a linear transformation of $\Sigma_i x_{ik}x_{il}$, which is the number of edges shared by paths k and l.

of walks that pass through a given node. We call these *medial* measures. The distinction between radial and medial measures forms the third classificatory dimension, which we call Walk Position.

The best known medial measure is Freeman's *betweenness centrality* (Freeman, 1980). Loosely described, the betweenness centrality of a node is the number of times that any actor needs a given actor to reach any other actor. A more precise definition is as follows: Let g_{ikj} denote the number of geodesic paths from node i to node j, and let g_{ikj} denote the number of geodesic paths from i to j that pass through intermediary k. Then the betweenness centrality is defined as follows:

$$C_k^{\mathrm{BET}} = \sum_i \sum_j \frac{g_{ikj}}{g_{ij}} \tag{10}$$

The measure is, in effect, k's share of all shortest-path traffic from i to j, summed across all choices of i and j. If there is only one shortest path from any point to any other, the measure is equal to the number of geodesic paths that pass through a given node k. One can easily imagine that when the network being studied consists of ties that are very costly to build, betweenness will indeed index an ability to extort benefits from flows through the network. For example, if the network represents trade routes between medieval cities, the cities with high betweenness centrality have opportunities for amassing wealth and exerting control that other cities would not have (Pitts, 1979).

Several variations on betweenness centrality are possible. First of all, the reliance on geodesic paths alone may be undesirable. While inter-city trade might well take only shortest paths (in order to minimize costs), information might flow equally well across all possible paths. Hence we would modify g_{ikj} to record the number of paths of any kind that link i and j via k. Borgatti (2002, 2005) considers betweenness for all possible paths, as well as all possible trails, as well as walks (weighted inversely by length). However, he uses simulation to estimate the betweenness values rather than formulas. Newman (2005) provides closed-form equations for the case of random traversal via walks.

Another set of variants is obtained by limiting the length of paths, on the idea that very long paths are seldom used and should not contribute to a node's betweenness. Such measures might be called *k-betweenness centrality*, where k gives the maximum length of paths counted. Friedkin (1991) proposes a measure that is essentially k-betweenness measures with $k = 2$. Similarly, Gould and Fernandez (1989) develop brokerage measures that are specific variants of 2-betweenness measures. A sophisticated variant would be a class of betweenness measures that count paths of all lengths, but weight them inversely in proportion to their length.

Of course, if we count all paths or walks we are in danger of double-counting since many paths can share the same subset of edges. If this is a concern, we would want to count only edge-disjoint paths. This is exactly what Freeman et al. (1991) have done. Their measure is called *flow betweenness* [11]. It is called flow betweenness because of the well-known relationship between the number of edge-independent paths between a pair of nodes, and the amount of material that can flow from one node to another via all possible edges (Ford and Fulkerson, 1956). Since flow betweenness assesses the proportion of edge-independent paths that involve a given node, it is in effect measuring the amount of flow in the network that would not occur if the node were not present (or were choosing not to transmit). This is really the essence of any betweenness measure: the potential for withholding flow, otherwise known as gatekeeping.

[11] Flow betweenness bears the same relationship to geodesic betweenness that information centrality bears to Freeman's closeness measure. Both information centrality and flow betweenness take into account all edge-disjoint paths, whereas Freeman's closeness and geodesic betweenness consider only geodesic paths.

An interesting aspect of flow betweenness is how it is computed. Because the sets of edge-independent paths between any two nodes are not unique, flow betweenness cannot be calculated directly by counting paths. Instead, programs like UCINET (Borgatti et al., 2002) essentially simulate the gatekeeping process by calculating flows between all pairs of nodes in the network, then removing the node whose centrality is being measured, and then recalculating the flows.

More specifically, let us denote by W the matrix of maximum flows between nodes (i.e., the number of edge-independent paths between them) and denote by ${}^kW$ the matrix derived from W by deleting row and column k. In addition, let us denote by ${}^kW^*$ the matrix obtained by deleting node k from the original network, and recalculating the flow matrix. We can then define flow centrality as follows:

$$c_k = \sum_{i,j} \frac{{}^k w_{i,j} - {}^k w^*_{i,j}}{{}^k w_{i,j}} \tag{11}$$

We can reformulate all betweenness measures in accordance with Eq. (11), simply by changing the W matrix. For example, to calculate Freeman's betweenness, we take W to be the geodesic count matrix in which w_{ij} gives the number of geodesic paths from i to j. Applying Eq. (11) gives scores exactly equal to twice Freeman's values.

Thus, betweenness-type measures might be thought of as "proportion reduction in cohesion" (PRC) measures, analogous to proportion reduction in error (PRE) measures in statistics. Where something flows over the links of a network, PRC measures quantify the potential of a node to disrupt flows throughout the network by ceasing its own transmissions.

If we define cohesion in terms of reachability, a PRC measure essentially indexes the network fragmentation that results from removing a node. Such a measure, called fragmentation (F), was introduced by Borgatti (2003, forthcoming). The F measure is simply the proportion of disconnected pairs of nodes that results when a given node is removed from a network. The bigger the value, the more important the node in maintaining cohesion. As implemented in UCINET (Borgatti et al., 2002), the measure is accompanied by a normalization in which the level of fragmentation of the original network is subtracted from the fragmentation after removing a given node, divided by the fragmentation of the original network. Large positive values indicate nodes that contribute to the cohesion of the network, while large negative values indicate nodes that reduce the cohesiveness of the network.

Defining cohesion in terms of distance yields a different set of measures. As a specific example, let us define a cohesion matrix W as the reciprocals of the geodesic distance between all pairs of nodes in a network (with the convention that the reciprocal of an undefined distance is 0). Now, we remove a given node (whose centrality we are measuring) from the W matrix, yielding ${}^kW$, and also remove the node from the original network and re-compute the reciprocals of geodesic distances among all pairs of remaining nodes, yielding ${}^kW*$. Then we apply Eq. (11), subtracting the values of ${}^kW*$ from the values of ${}^kW$ and dividing by ${}^kW$. Summing all of these adjusted values, we obtain the relative decrease in cohesion obtained by removing a given node.

A similar measure was proposed by Borgatti (2003, forthcoming). Called "distance-weighted fragmentation" (DF), this measure is defined as the average reciprocal distance among nodes after removal of a given node. The measure is 1 when all nodes are distance 1 from each other (i.e., a complete graph), and 0 when all nodes are isolates. Intermediate values index the extent to which the presence of a node tends to reduce distances in the network.

Table 1
Cross-classification of centrality measures

	Radial	Medial
Volume	Freeman degree, Sade k-path, Bonacich eigenvector, Katz status, Hubbell status, Hoede status, Doreian iterated, Hubbell, Markovsky et al. GPI, Friedkin TEC, Coleman power, Bonacich power, Burt prestige	Anthonisse rush, Freeman betweenness, Freeman et al. flow betweenness, Friedkin MEC, Newman RWB
Length	Freeman closeness, Stephenson-Zelen information, Friedkin IEC	Borgatti DF

4. A Typology of measures

It is apparent in this review of measures that all of the measures evaluate a node's involvement in the walk structure of a network. That is, they evaluate the volume or length of walks of some kind that originate, terminate, or pass through a node. Furthermore, all are based on the marginals of an appropriately constructed node-by-node matrix, although the method of calculating marginals can vary from simple sums to averages and weighted averages to harmonic means, and so on. Thus four basic dimensions distinguish between centrality measures: the types of walks considered (called Walk Type, such as geodesic or edge-disjoint), the properties of walks measured (called Walk Property, namely volume or length), the type of nodal involvement (called Walk Position, namely radial or medial), and type of summarization (called Summary Type, such as sum or average).[12]

The Walk Type dimension concerns the restrictions that some measures impose on the kind of walks considered, such as only geodesics, only true paths, limited length walks, and so on. The Walk Property dimension distinguishes between measures that evaluate the number of walks a node is involved in and measures that evaluate the length of those walks. The Walk Position dimension distinguishes between measures that evaluate walks emanating from a node and measures that evaluate walks passing through a node. The Summary Type dimension distinguishes measures using different ways to summarize rows of the walk matrix[13]. For simplicity, Table 1 gives a list of centrality measures cross-classified by just two of the dimensions: Walk Property (volume vs. length), and Walk Position (radial vs. medial).

While each centrality measure uses a different W matrix, in all cases the W matrix is an indicator of social proximity/cohesion among nodes.[14] Almost all of the W matrices we have seen have been identified specifically in the network literature as measures of cohesion. The i, jth cell of

[12] A fourth attribute, less important, is the choice of summary statistic. All centrality measures can be computed by summarizing the rows of an actor-by-actor matrix W (or A). Typically, this statistic is a simple sum or average. However, other summary statistics, such as weighted means, medians, modes, minimums and maximums, are used as well. This decision point is well known in cluster analysis, where the difference between some clustering methods is whether the minimum, maximum or median is used to compute the distance from a point to a set of points (Johnson, 1967; D'Andrade, 1978). An important example of a weighted mean is the eigenvector, which is used by Bonacich (1972), Burt (1982), and Doreian (1986).

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[14] We use the terms proximity and cohesion interchangeably to refer to social closeness or strength of connection between pairs of nodes.

any W matrix indicates the "connectedness" or "relatedness" of i and j, which is often assumed to correspond to a potential for transmission of attitudes, diseases, resources, etc. Of course, the values of W can be scaled such that large numbers indicated greater cohesion (as in a matrix of valued strengths of ties) or lesser cohesion (as in a geodesic distance matrix).

If the W matrix reflects dyadic cohesion, it is no surprise that summarizing the values of the W matrix gives the well-known measures of network cohesion such as density and characteristic path length. Density is simply the average of the adjacency matrix across all cells, while characteristic path length is the average (or other summary measure) of the geodesic distance matrix.

The cohesion matrix is also used as the basis for all methods of detecting cliques and other cohesive subgroups (e.g. Alba and Kadushin, 1976; Burt, 1991). The myriad definitions and techniques of cohesive subgroups can be viewed as identifying clusters within the cohesion matrix. Interestingly, the Walk Property dimension that we have identified for centrality measures is also a key dimension in the classification of cohesive subgroups (Borgatti et al., 1990). As is well-known, the cohesive subgroup known as a clique is characterized by having maximum density (number of ties) and minimum distance (length of paths). Since its formalization by Luce and Perry (1949), subsequent researchers have sought to relax the definition of clique by relaxing either the distances among members of the subgroup, or the number of ties (i.e., walks of length 1). Concepts such as n-cliques, n-clans, and n-clubs relax the minimal distance property of the clique, while concepts such as k-plexes, k-cores, ls-sets, and lambda-sets relax the maximum density property of cliques. In short: length versus volume.

Similarly, what centrality measures do is summarize the amount (and type) of dyadic cohesion that each node is involved in. In effect, centrality measures are indices of the share of dyadic cohesion attributable to each node. For comparison, whole network measures of cohesion, such as density or characteristic path length, are exactly like centrality measures except that instead of breaking out the summary by node, the entire cohesion matrix is summarized, much like the grand marginal in a contingency table.

Thus, a basic claim of the graph-theoretic typology presented here is that dyadic cohesion provides a common basis for not only centrality measures but subgroups and network cohesion. In this way, our analysis provides a sense of continuity with the other major areas of network analysis.

Another benefit that the typology provides is a partial answer to the commonly-asked question of how to choose among centrality measures. The typology essentially divides measures into groups that, to put it in marketing terms, are more competitive with each other than with other measures. Our claim is that measures within the same box in Table 1 are similar enough on key attributes that they can be thought of as competitive, i.e., as potentially substitutable alternatives for each other. Among measures within each box, we can reasonably ask which is better. In contrast, measures in different boxes differ in fundamental ways, and are perhaps best viewed as complementary.

5. Radial measures and the core periphery assumption

It is apparent that all radial measures are constructed the same way. First one defines an actor-by-actor matrix W that records the number or length of walks of some kind linking every pair of actors. Then one summarizes each row of W by taking some kind of mean or total. Thus, centrality provides an overall summary of a node's participation in the walk structure of the network. It is a measure of how much of the walk structure is due to a given node. It is quite literally the node's share of the total volume or length of walks in the network.

Thus, the essence of a radial centrality measure is this: radial centrality summarizes a node's connectedness with the rest of the network. To the extent that dyadic cohesion is seen as indexing influence (e.g., Katz, 1953; Hubbell, 1965; Friedkin, 1991), the centrality measure judges the overall influence of an actor. If the cohesion matrix is adjacency (as in degree centrality), and the adjacency matrix represents the "friend of" relation, then centrality summarizes an actor's "friendliness".

This raises a question. Under what conditions does it make sense to summarize, with a single value, a node's cohesion with all others? Consider the mean of any list of numbers. It can always be computed, but only serves as a summary when the distribution of the numbers is unimodal. Indeed, if the list is known to be normally distributed, the mean and standard deviation alone can generate the entire distribution. But if the shape of the distribution is bimodal, the mean is a very poor summary. If the ideal serving temperature of tea is in the range of 35–50° for much of the population (because they like iced tea) and the ideal serving temperature is in the range of 130–160° for the other half of the population (because they like hot tea), does the average of the ideal temperatures provide a good assessment of the population's tastes? That is, does a luke-warm temperature of 97.5° provide a good picture of what the people's tastes are? Probably not.

A radial centrality measure is clearly interpretable in a network in which dyadic cohesion is unimodal, but not in one which is multimodal. That is to say, radial centrality makes sense in networks which have, at most, one center. This means that a cohesive subgroup analysis would find only one subgroup (a core) to which all nodes belong to a greater or lesser extent. The network would not be divided in two or more subgroups. In that case, radial centrality would effectively serve as a measure of "coreness" (Borgatti and Everett, 1999; Everett and Borgatti, 2004), which is to say, strength of membership in the one and only group. Bonacich (1972) has noted that if a network contains more than one component (i.e., a maximal set of nodes that are mutually reachable), eigenvector centrality will assign zeros to all nodes not in the largest component, even if they are highly central in their own component. Rather, those nodes load highly on the remaining eigenvectors. In other words, the eigenvectors of a cohesion matrix measure strength of involvement of each node to each major subgroup (component).

Before we can interpret a radial measure of centrality, we must determine whether the network satisfies the one-group requirement. A network will exhibit a core-periphery pattern whenever its cohesion matrix can be modeled as a non-negative function of its marginals (Borgatti and Everett, 1999). For example, if the cohesion matrix W is a valued adjacency matrix in which w_{ij} gives the number of interactions observed between i and j, and the model of independence holds (i.e., no interaction), then the network has a core-periphery structure. This is because the independence model specifies that the extent of dyadic cohesion between nodes i and j is proportional to the product of their general proximity to anyone (i.e., their centrality). Hence, the only region of the network with high densities of proximity will be populated by high centrality nodes, and there will only be one such region. This pattern of distribution of proximities is precisely a core-periphery structure.

Interestingly, one measure of centrality "comes with" such a test built-in: Bonacich's eigenvector centrality. The eigenvectors and eigenvalues of any symmetric matrix can be multiplied to recreate the matrix, as shown in Eq. (12):

$$A = V'DV$$
$$a_{ij} = \sum_k v_{ik} e_k v_{jk} \qquad (12)$$

In the equation, V is an $n \times n$ matrix whose columns are the eigenvectors of A, the matrix D is a diagonal matrix of eigenvalues, and e rewrites those eigenvalues as a simple vector such that $e_k = d_{kk}$. If the summation is performed for all k ranging from 1 to n, the approximation is exact. If the summation is performed using fewer than n eigenvectors, the approximation is as close as possible under a least squares criterion. Since eigenvector centrality is defined as the eigenvector of A with the largest eigenvalue, it is the single best vector for estimating the values of A under the following simple model:

$$a_{ij} = \lambda_i^{\mathrm{EIG}} c_j^{\mathrm{EIG}} \tag{13}$$

According to the equation, the existence (or strength) of the tie between nodes i and j is approximated by the product of their centralities (adjusted by λ, which serves as a scaling constant). The accuracy of the approximation is roughly indexed by the size of the eigenvalue, relative to the others. If all other eigenvalues are near zero, the approximation will be nearly perfect. Thus, the model fits when the eigenvector centrality alone is sufficient to reproduce the observed pattern of ties. When this occurs, the network necessarily exhibits a core-periphery pattern[15]. The relative size of the largest eigenvalue can therefore be interpreted as indicating the extent to which the network has a core-periphery structure.

The fit of a core-periphery model to an observed network may be seen as a generalized measure of network centralization. Freeman (1979) defines centralization as the sum of differences between the centrality of the most central node and all other nodes, divided by the same sum calculated on a star graph with the same number of nodes. The idea is that the star epitomizes the ideal of a centralized network, and Freeman's statistic gives the extent to which the observed network conforms to the ideal type. It is, therefore, a measure of fit between a network and an ideal model. The difference is in the choice of ideals. The star is only one example of a network structured as a core and periphery. What about networks with more than one node in the core? What about networks with nodes that are neither central nor peripheral, but somewhere in between? The core-periphery model can include networks with all of these characteristics.

Since the generic formula for a radial centrality measure is

$$c_i = \sum_j w_{ij}, \tag{14}$$

radial centrality can be seen as a partitioning of total network cohesion ($\Sigma\Sigma w_{ij}$) by actor. The centrality of a node is its share of, or contribution to, total cohesion. If the number of nodes in the network is fixed, the total cohesion is a measure of overall cohesive density, and centrality is a node's contribution to that density.

However, in general we cannot view centrality as *generating* cohesion. The core-periphery model does not necessarily carry with it a *process* model. Consider, for example, fitting the independence model to a W matrix whose cells record the number of paths from each node to every other. If the model fits well, we are safe in concluding that the data form a core-periphery structure, and therefore a k-path centrality measure has a reasonable interpretation. But we have not specified a theory that explains how an underlying attribute of actors causes the observed pattern of paths. Such a theory is difficult to construct. If the cohesion matrix were simple adjacency for

[15] Of course, all nodes can have equal centrality and satisfy the model, in which case we might either choose to regard all nodes as core or all nodes as periphery. The big point is that such a model cannot generate a structure with two or more cores (i.e. subgroups).

the "friend of" relation, it is plausible that a property of actors (e.g. friendliness) might determine the probability of forming a tie with another actor (cf., Holland and Leinhardt, 1981). But it is difficult to understand how friendliness works to create a specific number of *paths* or walks, since walks are as much functions of all the other nodes as they are of the two endpoints. Thus most centrality measures should be thought of as summaries of a node's position in a one-group core-periphery structure, but not as parameters that generate that structure.

It should be noted that medial measures of centrality do not make the same one-group assumption. These measures correctly assign particularly high centrality scores to nodes serving as bridges between subgroups. However, it is still the case that it is difficult to interpret a given value of medial centrality without knowing the group's cohesive structure. For example, the center of a sociometric star (a core-periphery structure) is not only highly medial, it is also central in more conventional, radial, ways as well (i.e., it is "in the thick of things"). In contrast, the liaison between several different subgroups can be very high on a medial centrality measure, and yet be only peripheral to each subgroup.

Some empirical evidence of the fundamental difference between medial and radial measures may be found in a study by Nakao (1990). She computed Freeman's graph centralization measures on all possible graphs of 4, 5, 6, 7, and 8 nodes. After computing correlations among the measures for each size of network, she concludes: "These correlations show that the betweenness-based centrality measure behaves somewhat differently from the other two measures, indicated by lower correlations. The degree-based measure and the closeness-based measure are related very closely in a linear manner." She also notes that "The graphs whose order of centrality values is $C^{\mathrm{BET}} > C^{\mathrm{CLO}} > C^{\mathrm{DEG}}$ tend to be divided into sub-clusters that are connected to each other by a line or via a focal point. This type of graph may be characterized as a decentralized network, in the manner defined in organizational research. On the other hand, the graphs which produce $C^{\mathrm{CLO}} > C^{\mathrm{DEG}} > C^{\mathrm{BET}}$ would be described as one-cluster networks which contain a circle involving a large proportion of points in the network."

6. Discussion

The differences between radial and medial measures discussed in the last section suggest that this distinction is more important than the volume versus length distinction. In choosing between volume and length measures, one is choosing between different conceptions of cohesion. It seems plausible to suggest that, for a given theoretical application, it is possible to say that one is better than the other. For example, if one is studying risk of receiving in a timely manner something flowing through the network, it would seem that length measures make the most sense since they map directly to the expected arrival times (Borgatti, 1995). When the concern is with the certainty of arrival of something flowing through the network, volume measures would seem like an obvious choice.

In contrast, the choice between radial and medial measures can be seen in terms of the distinct roles played by nodes in the network. For example, consider a cohesion matrix W defined as the number of paths between all pairs of nodes. Thus, there exists an inventory of every path in the network. A radial measure of volume counts the number of these paths in which a given node serves as an endpoint. A medial measure counts the number of these paths in which the node serves as an interior point. Together, the radial and the medial add up to the total number of paths that a node is involved with in any role. In this sense, we can speak of decomposing a node's total involvement in the paths of a network into radial and medial portions, as shown in Eq. (15). If so, radial and medial measures are complementary and both are needed to deliver a complete picture of a node's contribution to the network (cf., Friedkin, 1991). Whereas radial measures assess group membership, medial measures assess bridging, reminiscent of the distinction in the social capital literature of bonding social capital and bridging social capital, or closed versus open ego networks.

$$\text{Total Involvement} = \text{Radiality} + \text{Mediality} \tag{15}$$

It should be noted that, for most of the field of centrality, this decomposition is metaphorical rather than literal. It is literally accurate for the specific example given, but not elsewhere. For example, while nodes can only be in one position on a path, they can occur in multiple positions in a trail or walk, so that radial and medial counts no longer add to the total number of sequences. In addition, measures like Freeman's betweenness do not just count the number of times a node occupies an interior position of a geodesic, but weight those times according to exclusivity.

7. Conclusion

Following Sabidussi (1966), we have described the notion of centrality in purely graph-theoretic terms: what all measures of centrality do is assess a node's involvement in the walk structure of a network. This is the graph-theoretic answer to the question 'What do centrality measures measure?' We have suggested that centrality measures differ along four key dimensions: choice of summary measure, type of walk considered, property of walk assessed, and type of involvement. The choice of summary dimension has the least variance, consisting mostly of simple sums and averages, along with a few exemplars of weighted sums (e.g., eigenvectors) and centroids. The type of walk dimension distinguishes measures based on edges, geodesics, paths, trails and walks. The property of walk dimension distinguishes between volume and length measures. The type of involvement dimension distinguishes between radial and medial measures.

It can be seen that the single distinction made by Borgatti (2005) between frequency (measuring how often something flows across a node) and time (how soon something flows to a node) can be derived as a collapsing of the property of walk and type of involvement dimensions. That is, the frequency-based measures in Borgatti (2005) are medial-volume measures in the present terminology while the time-based measures correspond to radial-length measures. In addition, the cross-classification of measures by type of involvement and property of walk results in a four-fold classification that is not inconsistent with Freeman's (1979) three-fold categorization.

Radial measures in particular are reductions via aggregation operators of pairwise proximities to attributes of nodes or actors. These aggregations range from simple marginals (degree) to weighted marginals (eigenvectors) to distances along euclidean axes (centroid). The types of reductions correspond to standard statistical scaling and modeling techniques: simple marginals correspond to fitting the log-linear model of quasi-independence to a square, symmetric table with missing diagonals; eigenvectors correspond to factoring a correlation or covariance matrix; and euclidean axes correspond to a one-dimensional MDS scaling of a cohesion matrix. Not surprisingly, the usefulness and interpretability of radial measures depends on the fit of the cohesion matrix to the one-dimensional model, just as in univariate statistics the mean is most interpretable when applied to a unimodal distribution.

As discussed by Borgatti and Everett (1999), a network that is fit by a one-dimensional model has a core-periphery structure in which all nodes revolve more or less closely around a single core. Thus, radial centrality measures are most interpretable when the cohesion matrix passes a test of core-peripheriness, in which case the measures can be viewed as measures of "coreness".

Just as radial measures were shown to largely reduce to marginals of a cohesion matrix W, medial measures were also reduced to a common formulaic structure that we referred to as proportion reduction in cohesion. As such, medial measures essentially measure the impact of the presence of a node on the dyadic cohesion among all pairs of nodes. In other words, they measure the change in cohesion that would result from removing a given node. As such, medial measures do not depend on core/periphery structures for interpretability, and in fact are particularly useful when networks have "clumpy" structures characterized by wide variation in local density.

At a general level, we note the relationship of centrality concepts with the concepts of graph cohesion and cohesive subgroups. The key underlying concept is that of dyadic cohesion—the social proximity of pairs of actors in a network. Dyadic cohesion is what is measured by the W matrix that undergirds all measures of centrality. There are two fundamental ways of analyzing cohesion. One is to seek regions of the network that are more cohesive than others—a focus on the pattern of cohesion. This constitutes the field of cohesive subgroups. The other is to attribute to individual nodes their share of responsibility for the cohesion of the network—a focus on the amount of cohesion. This constitutes the field of centrality measures. Within that, two fundamental approaches are discernable—the radial approach that directly partitions total cohesion by node, and the medial approach that assesses a node's contribution to cohesion by removing it. The other fundamental distinction—between volume and length measures—is essentially an argument about the meaning of cohesion.

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